

# T1 CRITERIONS FOR GENERALISED CALDERÓN–ZYGmund TYPE OPERATORS ON HARDY AND BMO SPACES ASSOCIATED TO SCHRÖDINGER OPERATORS AND APPLICATIONS

THE ANH BUI, JI LI, AND FU KEN LY

ABSTRACT. Suppose  $L = -\Delta + V$  is a Schrödinger operator on  $\mathbb{R}^n$  with a potential  $V$  belonging to certain reverse Hölder class  $RH_\sigma$  with  $\sigma \geq n/2$ . The main aim of this paper is to provide necessary and sufficient conditions in terms of  $T1$  criteria for a generalised Calderón–Zygmund type operator with respect to  $L$  to be bounded on Hardy spaces  $H_L^p(\mathbb{R}^n)$  and on BMO type spaces  $BMO_L^\alpha(\mathbb{R}^n)$  associated with  $L$ . As applications, we prove the boundedness for several singular integral operators associated to  $L$ . Our approach is flexible enough to prove the boundedness of the Riesz transforms related to  $L$  with  $n/2 \leq \sigma < n$  which were investigated in [22] under the stronger condition  $\sigma \geq n$ . Thus our results not only recover existing results in [22] but also contains new results in literature.

## CONTENTS

1. Introduction	1
1.1. Main results	2
1.2. Applications	4
2. Preliminaries	6
3. Hardy and Campanato spaces associated with Schrödinger operator	7
3.1. Hardy spaces	7
3.2. Molecular characterizations	9
3.3. Campanato spaces	13
4. Proof of the $T1$ criterions for $H_L^p(\mathbb{R}^n)$ and $BMO_L^\alpha(\mathbb{R}^n)$	14
5. Proofs of Applications	18
5.1. Laplace transform type multipliers	18
5.2. Riesz transforms $\nabla L^{-1/2}$ and $\nabla^2 L^{-1}$	19
5.3. Riesz transforms $V^s L^{-s}$ , $0 < s \leq 1$ , and their adjoints	20
References	25

## 1. INTRODUCTION

It is well-known that the  $T1$  theorem plays a crucial role in the analysis of  $L^2$  boundedness (and furthermore the  $L^p$  boundedness) of Calderon–Zygmund singular integral operators (see [6] and [13, p. 590]). For the endpoint boundedness (i.e.  $p = 1$  and  $p = \infty$ ), there are also analogous  $T1$  criterions for Calderón–Zygmund operators. To be more precise, suppose  $T$  is a Calderón–Zygmund operator (in the sequel we denote this by  $T \in CZO$ ), then  $T$  is bounded on the Hardy space  $H^1(\mathbb{R}^n)$  if and only if  $T^*1 = 0$ , and bounded on the BMO space  $BMO(\mathbb{R}^n)$  if and only if  $T1 = 0$  (see for example [15]).

---

2010 *Mathematics Subject Classification*: 35J10, 42B20, 42B30, 42B35.

*Key words and phrases*:  $T1$  criterion, Hardy space, BMO space, Schrödinger operators, reverse Hölder class, Riesz transforms, Laplace transform type multipliers.

Recently, Betancor et al. [2] established a  $T1$  criterion for Hermite–Calderón–Zygmund operators on the BMO space  $BMO_H(\mathbb{R}^n)$  associated to the Hermite operator (also known as harmonic oscillator)  $H = -\Delta + |x|^2$  in  $\mathbb{R}^n$ . Based on this criterion they studied systematically the boundedness of certain singular integral operators related to  $H$  on  $BMO_H(\mathbb{R}^n)$ , such as Riesz transforms, maximal operators related to the heat and Poisson semigroups, Littlewood–Paley  $g$ -functions, as well as variation operators. This  $T1$  criterion was generalised by Ma et al. [22], where they established a  $T1$  criterion for boundedness in the Campanato type spaces  $BMO_L^\alpha(\mathbb{R}^n)$  of so-called  $\gamma$ -Schrödinger–Calderón–Zygmund operators, which are related to the Schrödinger operator  $L$  on  $\mathbb{R}^n$ ,  $n \geq 3$ , given by

$$(1.1) \quad L = -\Delta + V, \quad V \in RH_\sigma, \quad \sigma \geq n/2.$$

The expression  $V \in RH_\sigma$  means that  $V$  is a non-negative function that satisfies the reverse Hölder inequality

$$(1.2) \quad \left( \frac{1}{|B|} \int_B V(y)^\sigma dy \right)^{\frac{1}{\sigma}} \leq \frac{C}{|B|} \int_B V(y) dy.$$

for some constant  $C = C(q, V)$  and every ball  $B$ .

As applications, they obtained regularity estimates for certain operators related to  $L$  such as the maximal operators and square functions of the heat and Poisson semigroups, for Laplace transform type multipliers, for negative powers  $L^{-\gamma/2}$ . Moreover, on restricting  $\sigma \geq n$ , they obtained regularity estimates for the Riesz transforms  $\nabla L^{-1/2}$ .

Shen [23] proved that when  $\sigma \geq n$ , the Riesz transforms  $\nabla L^{-1/2}$  are Calderón–Zygmund operators. However, this may not be true when  $n/2 \leq \sigma < n$  because pointwise estimates on the kernel of  $\nabla L^{-1/2}$  are not available. But certain weaker estimates related to the standard Hörmander condition

$$(1.3) \quad \int_{|x-y|>\delta|y-\bar{y}|} |K(x, y) - K(x, \bar{y})| dx \leq C$$

have been derived in [4, 14], for some  $C > 0$  and  $\delta > 1$  and every  $y, \bar{y} \in \mathbb{R}^n$ .

The aim of this article is to provide necessary and sufficient conditions for a larger class of generalised Calderón–Zygmund type operator  $T$  to be bounded on  $H_L^p(\mathbb{R}^n)$ , where  $L = -\Delta + V$  is a Schrödinger operator with  $V \in RH_\sigma$  for some  $\sigma \geq n/2$ . The conditions are phrased as conditions on the object  $T^*1$ . As a consequence we also obtain the criterion for such operators  $T$  to be bounded on  $BMO_L^\alpha(\mathbb{R}^n)$ , with conditions phrased on  $T1$ . We would like to describe briefly our contributions in this paper.

- (i) Unlike [22], we do not assume pointwise and smoothness conditions on the associated kernel of our generalised Calderón–Zygmund type operators  $T$ . This allows us to relax the condition  $\sigma \geq n$  when considering the Riesz transforms  $\nabla L^{-1/2}$ , and also allows us to consider such operators as  $V^{1/2}L^{-1/2}$  and  $VL^{-1}$ .
- (ii) Our results recover those in [2] for the Hermite–Calderón–Zygmund operators, and those in [22] for their  $\gamma$ -Schrödinger–Calderón–Zygmund operators  $T$  when  $\gamma = 0$ .
- (iii) The result for boundedness on Hardy spaces (Theorem 1.2) is new in the literature.
- (iv) To prove the boundedness on Hardy spaces, we introduce an  $L$ -molecule satisfying size and weak cancellation condition, which is different from the  $L$ -molecules in the direction of work in [1, 7, 16]. Then we establish the molecular characterization of Hardy spaces.

**1.1. Main results.** In the sequel we set  $L$  as in (1.1).

The critical radius function (introduced by Shen [23]) associated to the potential  $V \in RH_\sigma$  with  $\sigma \geq n/2$  is defined by

$$(1.4) \quad \rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}.$$

As an example for the harmonic oscillator with  $V(x) = |x|^2$ , we have  $\rho(x) \sim (1 + |x|)^{-1}$ .

We also set  $\sigma_0 := 2 - n/\sigma$ , a constant which will play a key role in this article. Note that  $0 < \sigma_0 \leq 1$  precisely when  $\frac{n}{2} < \sigma \leq n$ .

We now introduce generalized Calderón–Zygmund type operators with respect to  $L$  defined in (1.1) as follows.

**Definition 1.1.** *Let  $\gamma > 0$ ,  $1 < \theta < \infty$  and  $\theta'$  be the conjugate of  $\theta$ . We say that  $T \in GCZK_\rho(\gamma, \theta)$  if  $T$  has an associated kernel  $K(x, y)$  satisfying the following estimates:*

(i) *For each  $N > 0$  there is a constant  $C_N > 0$  such that*

$$(1.5) \quad \left( \int_{R < |x - x_B| < 2R} |K(x, y)|^\theta dx \right)^{1/\theta} \leq C_N R^{-n/\theta'} \left( \frac{\rho(x_B)}{R} \right)^N$$

*for all  $y \in B(x_B, \rho(x_B))$  and all  $R > 2\rho(x_B)$ .*

(ii) *There is a constant  $C > 0$  such that*

$$(1.6) \quad \left( \int_{2^k r_B < |x - x_B| < 2^{k+1} r_B} |K(x, y) - K(x, x_B)|^\theta dx \right)^{1/\theta} \leq C 2^{-k\gamma} |2^k B|^{-1/\theta'}$$

*for all balls  $B = B(x_B, r_B)$ , all  $y \in B$  and  $k \geq 1$ .*

*We say that  $T \in GCZO_\rho(\gamma, \theta)$  if  $T \in GCZK_\rho(\gamma, \theta)$  and  $T$  is bounded on  $L^\theta(\mathbb{R}^n)$ .*

Note that the condition (1.6) implies the standard Hömander condition (1.3), and therefore, if  $T \in GCZO_\rho(\gamma, \theta)$  for some  $\gamma$  and  $\theta$ , then  $T$  is of weak type  $(1, 1)$  and hence is bounded on  $L^p$  for all  $1 < p \leq \theta$ .

We point out that the Hermite–Calderón–Zygmund operators of [2] and the  $\gamma$ -Schrödinger–Calderón–Zygmund operators  $T$  when  $\gamma = 0$  of [22] belong to  $GCZO_\rho(\delta, \theta)$  for certain  $\delta$  and any  $1 < \theta < \infty$ .

It is well known that in the classical situation (see [15] for example) if  $T \in CZO$  then  $T$  is bounded on the Lipschitz  $\Lambda^\alpha$  for  $0 < \alpha < \gamma \leq 1$  if and only if  $T1$  is constant (we note that the Lipschitz spaces  $\Lambda^\alpha$  coincide with the Campanato spaces  $BMO^\alpha$  [5]). However, for Calderón–Zygmund type operators  $T$  with respect to Schrödinger operators  $L$ , there exist certain operators  $T$  for which  $T1$  or  $T^*1$  is non-constant. Notable examples are the Riesz transforms  $T = \nabla L^{-1/2}$ .

Our main result is the following  $T1$  type theorem for  $T \in GCZO_\rho(\gamma, \theta)$  to be bounded on Hardy spaces  $H_L^p(\mathbb{R}^n)$  associated with  $L$  defined in (1.1). For the precise definition and the properties of  $H_L^p(\mathbb{R}^n)$  we refer to Section 3.1.

**Theorem 1.2.** *Let  $T \in GCZO_\rho(\gamma, \theta)$  for some  $0 < \gamma < \sigma_0$ , where  $\sigma_0 := 2 - n/\sigma$ . Then:*

(a)  *$T$  is bounded on  $H_L^1(\mathbb{R}^n)$  if and only if  $T^*1$  satisfies*

$$\log \left( \frac{\rho(x_B)}{r_B} \right) \left( \frac{1}{|B|} \int_B |T^*1(y) - (T^*1)_B|^{\theta'} dy \right)^{1/\theta'} \leq C$$

*for every ball  $B$  with  $r_B \leq \frac{1}{2}\rho(x_B)$ .*

(b) *If  $\frac{n}{n+\gamma} < p < 1$ , then  $T$  is bounded on  $H_L^p(\mathbb{R}^n)$  if and only if  $T^*1$  satisfies*

$$\left( \frac{\rho(x_B)}{r_B} \right)^{n(1/p-1)} \left( \frac{1}{|B|} \int_B |T^*1(y) - (T^*1)_B|^{\theta'} dy \right)^{1/\theta'} \leq C$$

*for every ball  $B$  with  $r_B \leq \frac{1}{2}\rho(x_B)$ .*

(c) *If  $\frac{n}{n+\gamma} < p \leq 1$ , then  $T$  is bounded from  $H_L^p(\mathbb{R}^n)$  to the classical Hardy space  $H^p(\mathbb{R}^n)$  if and only if  $T^*1 = 0$ .*

Note that for  $\frac{n}{n+\sigma_0 \wedge 1} < p \leq 1$ , the cancellation condition for atoms in Definition 3.2 imply that the classical Hardy spaces  $H^p(\mathbb{R}^n)$  are strictly contained in  $H_L^p(\mathbb{R}^n)$ , and thus Theorem 1.2 also gives boundedness from  $H^p(\mathbb{R}^n)$  into  $H_L^p(\mathbb{R}^n)$  for (a), (b), and into  $H^p(\mathbb{R}^n)$  for (c).

The strategy of our proof of Theorem 1.2 proceeds in two steps. We firstly characterize  $H_L^p(\mathbb{R}^n)$  in terms of molecules associated with  $L$  that have certain size and cancellation conditions (different to the  $L$ -molecules in the direction of work in [1, 7, 16]). See Definitions 3.6 and 3.7. Secondly we show that the operators satisfying the conditions in Theorem 1.2 map atoms into molecules, which yields their boundedness on  $H_L^p(\mathbb{R}^n)$ .

As a consequence of Theorem 1.2 and the duality of the Hardy space  $H_L^p(\mathbb{R}^n)$  with BMO type spaces (also known as the Campanato space)  $BMO_L^\alpha(\mathbb{R}^n)$ , we obtain directly a T1 criterion for  $BMO_L^\alpha(\mathbb{R}^n)$  which extends the results of [2, 22] to a more general setting. For the precise definition and the properties of  $BMO_L^\alpha(\mathbb{R}^n)$  we refer to Section 3.3.

**Definition 1.3.** Let  $\gamma > 0$ ,  $1 < \theta < \infty$  and  $\theta'$  be the conjugate of  $\theta$ . We say that  $T \in GCZK_\rho^*(\gamma, \theta')$  if  $T$  has an associated kernel  $K(x, y)$  satisfying the following estimates:

(i)' For each  $N > 0$  there is a constant  $C_N > 0$  such that

$$(1.7) \quad \left( \int_{R < |y-x_B| < 2R} |K(x, y)|^{\theta'} dy \right)^{1/\theta'} \leq C_N R^{-n/\theta} \left( \frac{\rho(x_B)}{R} \right)^N$$

for all  $x \in B(x_B, \rho(x_B))$  and all  $R > 2\rho(x_B)$ .

(ii)' There are constants  $0 < \gamma \leq 1$  and  $C > 0$  such that

$$(1.8) \quad \left( \int_{2^k r_B < |y-x_B| < 2^{k+1} r_B} |K(x, y) - K(x_B, y)|^{\theta'} dy \right)^{1/\theta'} \leq C 2^{-k\gamma} |2^k B|^{-1/\theta}$$

for all balls  $B = B(x_B, r_B)$ , all  $x \in B$  and  $k \geq 1$ .

We say that  $T \in GCZO_\rho^*(\gamma, \theta')$  if  $T \in GCZK_\rho^*(\gamma, \theta')$  and  $T$  is bounded on  $L^{\theta'}(\mathbb{R}^n)$ .

We wish to make two observations. Firstly, whereas Definition 1.1 specifies a certain regularity in the second variable, the requirement here is in the first variable. Secondly if  $T$  belongs to  $GCZO_\rho^*(\gamma, \theta')$  for some  $\gamma$  and  $\theta$ , then  $T$  is automatically bounded on  $L^p$  for all  $\theta' \leq p < \infty$ .

**Theorem 1.4.** Let  $T \in GCZO_\rho^*(\gamma, \theta')$  for some  $0 < \gamma < \sigma_0$ , where  $\sigma_0 := 2 - n/\sigma$ . Then:

(a)'  $T$  is bounded on  $BMO_L(\mathbb{R}^n)$  if and only if T1 satisfies

$$\log \left( \frac{\rho(x_B)}{r_B} \right) \left( \frac{1}{|B|} \int_B |T1(y) - (T1)_B|^\theta dy \right)^{1/\theta} \leq C$$

for every ball  $B$  with  $r_B \leq \frac{1}{2}\rho(x_B)$ .

(b)' If  $0 < \alpha < \gamma$ , then  $T$  is bounded on  $BMO_L^\alpha(\mathbb{R}^n)$  if and only if T1 satisfies

$$\left( \frac{\rho(x_B)}{r_B} \right)^\alpha \left( \frac{1}{|B|} \int_B |T1(y) - (T1)_B|^\theta dy \right)^{1/\theta} \leq C$$

for every ball  $B$  with  $r_B \leq \frac{1}{2}\rho(x_B)$ .

(c)' If  $0 < \alpha < \gamma$  then  $T$  is bounded from  $BMO^\alpha(\mathbb{R}^n)$  into  $BMO_L^\alpha(\mathbb{R}^n)$  if and only if  $T1 = 0$ .

**1.2. Applications.** We now present some applications to singular integrals related to  $L$ . The precise definitions of the listed operators will be provided in Section 5.1.

**Theorem 1.5.** For  $\frac{n}{n+\sigma_0 \wedge 1} < p \leq 1$ , the Laplace transform type multipliers  $m(L)$  are bounded on  $H_L^p(\mathbb{R}^n)$ . As a consequence, for  $0 \leq \alpha < \sigma_0 \wedge 1$ , these operators are bounded on  $BMO_L^\alpha(\mathbb{R}^n)$ .

We point out that the above result recovers the  $BMO_L^\alpha$  result in Theorem 1.3 in [22], while the hardy space result is new. We also mention that using the vector-valued approach in [22], we can also apply Theorem 1.4 to recover boundedness on  $BMO_L^\alpha$  of the other operators listed in [22] Theorem 1.3, namely the maximal operators and Littlewood–Paley  $g$ -functions associated with the heat and Poisson semigroups.

Next we have the following result for the Riesz transforms  $\mathcal{R}_{(1)} = \nabla L^{-1/2}$  and  $\mathcal{R}_{(2)} = \nabla^2 L^{-1}$ .

**Theorem 1.6.** *The Riesz transforms  $\mathcal{R}_{(1)}$  and  $\mathcal{R}_{(2)}$  are bounded from  $H_L^p(\mathbb{R}^n)$  into  $H^p(\mathbb{R}^n)$  for all  $\frac{n}{n+\sigma_0 \wedge 1} < p \leq 1$ . As a consequence  $\mathcal{R}_{(1)}^*$  and  $\mathcal{R}_{(2)}^*$  are bounded from  $BMO^\alpha(\mathbb{R}^n)$  to  $BMO_L^\alpha(\mathbb{R}^n)$  for  $0 \leq \alpha < \sigma_0 \wedge 1$ .*

The results in Theorem 1.6 are not new. Indeed it is known that both  $\mathcal{R}_{(1)}$  and  $\mathcal{R}_{(2)}$  are bounded from  $H_L^p$  into  $L^p$  for all  $0 < p \leq 1$  and from  $H_L^p$  into  $H^p$  for all  $\frac{n}{n+1} < p \leq 1$  (see [16, 17, 20]).

We also apply our results to Riesz transforms induced by the potential  $V$  such as  $V^{1/2}L^{-1/2}$  and  $VL^{-1}$ , which were earlier shown by Shen [23] to be  $L^p$ -bounded for  $1 \leq p \leq 2\sigma$  and  $1 \leq p \leq \sigma$  respectively. While such operators are not of Calderón–Zygmund type, we will see that they nonetheless fall into the scope of Theorems 1.2 and 1.4.

In fact we shall consider their generalizations  $V^s L^{-s}$ , for  $0 < s \leq 1$ , which are  $L^p$  bounded for  $1 < p < \frac{2}{s}$  (see [25]).

**Theorem 1.7.** *For each  $0 < s \leq 1$  the operators  $V^s L^{-s}$  are bounded on  $H_L^p(\mathbb{R}^n)$  for each  $\frac{n}{n+s\sigma_0 \wedge 1} < p \leq 1$ . As a consequence the operators  $(V^s L^{-s})^*$  are bounded on  $BMO_L^\alpha$  for each  $0 \leq \alpha < s\sigma_0 \wedge 1$ .*

The results in Theorem 1.7 are new, although the cases  $s = \frac{1}{2}$  and  $s = 1$  are known to map  $H_L^p$  into  $L^p$  for  $\frac{n}{n+1} < p \leq 1$  (see [20]).

One may ask which operators  $T$  and their adjoints  $T^*$  are both bounded on  $H_L^p$  (and consequently  $BMO_L^\alpha$ )? Applying Theorems 1.2 and 1.4 would require that they be members of both  $GCZO$  and  $GCZO^*$ , and recall from earlier remarks that this imposes the  $L^p$  boundedness of  $T$  for  $p$  close to both 1 and  $\infty$ . This can be guaranteed for example when  $T$  is a Calderón–Zygmund operator, which is true of  $\mathcal{R}_{(1)}$  when  $\sigma \geq n$ , and of  $\mathcal{R}_{(2)}$  when  $V$  is a non-negative polynomial [33]. In our final application, we show that with sufficient regularity on  $V$ , the operators  $V^s L^{-s}$  and their adjoints  $L^{-s} V^s$  both fall into the scope of Theorem 1.2.

**Theorem 1.8.** *Suppose that  $V \in RH_\infty$  and that for some  $C > 0$*

$$(1.9) \quad |\nabla V(x)| \leq C\rho(x)^{-3} \quad \text{a.e. } x$$

*Then for each  $0 < s \leq 1$ , the operator  $L^{-s} V^s$  is bounded from  $H_L^p$  into  $H_L^p$  for  $\frac{n}{n+2s \wedge 1} < p \leq 1$ . As a consequence  $V^s L^{-s}$  is bounded from  $BMO_L^\alpha$  into  $BMO_L^\alpha$  for all  $0 \leq \alpha < 2s \wedge 1$ .*

The condition  $V \in RH_\infty$  ensures that both  $V^s L^{-s}$  and  $L^{-s} V^s$  are  $L^p$  bounded for all  $1 < p < \infty$ , while (1.9) furnish sufficient smoothness for the conditions of Theorems 1.2 and 1.4 to hold. Examples of  $V$  satisfying the conditions of Theorem 1.8 are non-negative polynomials and in particular include the harmonic oscillator  $V(x) = |x|^2$ .

This paper is organised as follows. In Section 3 we recall the Hardy and BMO type spaces associated to Schrödinger operator  $L$ , and introduce a new molecular decomposition for the Hardy spaces. In Section 4 we provide the proof of the  $T1$  criterions Theorems 1.2 and 1.4

for Hardy and BMO type spaces respectively. Finally in Section 5 we give applications of the  $T1$  criterion by proving Theorems 1.5–1.8.

Throughout the paper, we always use  $C$  and  $c$  to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We will write  $A \lesssim B$  if there is a universal constant  $C$  so that  $A \leq CB$  and  $A \sim B$  if  $A \lesssim B$  and  $B \lesssim A$ . Given a ball  $B$  we refer to the ball  $B(x_B, r_B)$  with centre  $x_B$  and radius  $r_B$ . We also denote by  $\rho_B := \rho(x_B)$ . The notation

$$\oint_B f = \frac{1}{|B|} \int_B f$$

refers to the average of  $f$  on  $B$ . The expression  $a \wedge b$  denotes the minimum of  $a$  and  $b$ . Given a ball  $B$ , the set  $U_j(B)$  denotes  $2^j B \setminus 2^{j-1} B$  for  $j \geq 1$  and denotes  $B$  if  $j = 0$ .

## 2. PRELIMINARIES

In this section we recall the well-known heat kernel upper bounds for the Schrödinger operator as well as properties for  $V$  and its critical radius function  $\rho$  as defined in (1.4).

The following estimates on the heat kernel of  $L$  are well known.

**Proposition 2.1.** ([11, 12]) *Let  $L = -\Delta + V$  with  $V \in RH_\sigma$  for some  $\sigma \geq n/2$ . Then for each  $N > 0$  there exists  $C_N > 0$  such that*

$$(2.1) \quad p_t(x, y) \leq C_N \frac{e^{-|x-y|^2/ct}}{t^{n/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}$$

and

$$(2.2) \quad |p_t(x, y) - p_t(x', y)| \leq C_N \left(\frac{|x - x'|}{\sqrt{t}}\right)^{\sigma_1} \frac{e^{-|x-y|^2/ct}}{t^{n/2}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}$$

whenever  $|x - x'| \leq \sqrt{t}$  and for any  $0 < \sigma_1 < \sigma_0$ .

For  $\sigma > 1$ , the class of locally integrable functions satisfying (1.2) will be denoted  $RH_\sigma$ . For  $\sigma = \infty$ , the left hand side of (1.2) is replaced by the essential supremum over  $B$ . It is well known that elements of  $RH_\sigma$  are doubling measures, and that  $RH_\sigma \subset RH_{\sigma'}$  whenever  $\sigma' < \sigma$ .

We list but do not prove the following properties of the critical function  $\rho$  in [23].

**Lemma 2.2.** *Let  $\rho$  be the critical radius function associated with  $L$  defined in (1.4). Then we have:*

(i) *There exist positive constants  $k_0 \geq 1$  and  $C_0 > 0$  so that*

$$C_0^{-1} [\rho(x)]^{1+k_0} [\rho(x) + |x - y|]^{-k_0} \leq \rho(y) \leq C_0 [\rho(x)]^{1/(1+k_0)} [\rho(x) + |x - y|]^{k_0/(1+k_0)},$$

for all  $x, y \in \mathbb{R}^n$ .

*In particular for any ball  $B$ , and any  $x, y \in B$  then  $\rho(x) \leq C_0^2 \left(1 + \frac{r_B}{\rho_B}\right)^2 \rho(y)$ .*

(ii) *There exists  $C > 0$  so that*

$$\frac{1}{r^{n-2}} \int_{B(x, r)} V(y) dy \leq C \left(\frac{r}{R}\right)^{\sigma_0} \frac{1}{R^{n-2}} \int_{B(x, R)} V(y) dy$$

for all  $x \in M$  and  $R > r > 0$ .

(iii) *For any  $x \in M$ , we have*

$$\frac{1}{\rho(x)^{n-2}} \int_{B(x, \rho(x))} V(y) dy = 1.$$

(iv) There exists  $C > 0$  so that for any  $r > \rho(x)$

$$r^2 \int_{B(x, \rho(x))} V(y) dy \leq C \left( \frac{r}{\rho(x)} \right)^{n_0 - n + 2}$$

where  $n_0$  is the doubling order of  $V$ . That is,  $\int_{2B} V \lesssim 2^{n_0} \int_B V$  for any ball  $B$ .

**Remark 2.3.** It follows from Lemma 2.2 (ii) and (iii) that for any ball  $B$ ,

$$r_B^2 \int_B V(y) dy \lesssim \begin{cases} \left( \frac{r_B}{\rho_B} \right)^{\sigma_0} & r_B \leq \rho_B \\ \left( \frac{r_B}{\rho_B} \right)^{n_0 + 2 - n} & r_B > \rho_B \end{cases}$$

**Lemma 2.4** ([9]). Let  $\rho$  be a critical function associated to Schrödinger operators  $L = -\Delta + V$ . Then there exists a sequence of points  $\{x_\alpha\}_{\alpha \in \mathcal{I}} \subset \mathbb{R}^n$  and a family of functions  $\{\psi_\alpha\}_{\alpha \in \mathcal{I}}$  satisfying for some  $C > 0$

- (i)  $\bigcup_{\alpha} B(x_\alpha, \rho(x_\alpha)) = \mathbb{R}^n$ .
- (ii) For every  $\lambda \geq 1$  there exist constants  $C$  and  $N_1$  such that  $\sum_{\alpha} \chi_{B(x_\alpha, \rho(x_\alpha))} \leq C \lambda^{N_1}$ .
- (iii)  $\text{supp } \psi \subset B(x_\alpha, \rho(x_\alpha)/2)$  and  $0 \leq \psi_\alpha(x) \leq 1$  for all  $x \in \mathbb{R}^n$ ;
- (iv)  $|\psi_\alpha(x) - \psi_\alpha(y)| \leq C|x - y|/\rho(x_\alpha)$ ;
- (v)  $\sum_{\alpha} \psi_\alpha(x) = 1$  for all  $x \in \mathbb{R}^n$ .

### 3. HARDY AND CAMPANATO SPACES ASSOCIATED WITH SCHRÖDINGER OPERATOR

In this section we recall the definition of Hardy space  $H_L^p(\mathbb{R}^n)$  associated to  $L$  in terms of the maximal operator and of atoms. Then we introduce a new kind of molecule for these  $H_L^p(\mathbb{R}^n)$  in terms of size condition and weak cancellation condition, and then we provide the molecule characterisation for  $H_L^p(\mathbb{R}^n)$ . We also recall the BMO type space associated to  $L$ , and note that it is the dual of  $H_L^p(\mathbb{R}^n)$ .

**3.1. Hardy spaces.** We now recall some properties related to the atomic decomposition of Hardy spaces associated to Schrödinger operators. For further details on the theory of Hardy spaces associated to Schrödinger operators, we refer the reader to [10, 11, 12, 18, 28, 32] and the references therein.

We first define the maximal operator associated to the heat semigroup:

$$\mathcal{M}_L f(x) := \sup_{t>0} |e^{-tL} f(x)|$$

For  $0 < p \leq 1$  we denote by  $L_b^p(\mathbb{R}^n)$  the set of all  $L^p$ -functions with bounded support. We then set

$$\mathfrak{S}_p(\mathbb{R}^n) := \{f : f \in L_b^s(\mathbb{R}^n) \text{ for every } s \in [1, \infty]\}$$

Following [11] we define

**Definition 3.1** (Hardy spaces). For  $p \in (0, 1]$ , the Hardy space  $H_L^p(\mathbb{R}^n)$  is defined as the completion of

$$\mathbb{H}_L^p := \{f \in \mathfrak{S}_p : \mathcal{M}_L f \in L^p\}$$

in the quasi norm  $\|f\|_{H_L^p} := \|\mathcal{M}_L f\|_p$ .

**Definition 3.2.** Let  $0 < p \leq 1$  and  $1 < q \leq \infty$ . A function  $a$  is called an  $(p, q)_L$ -atom for  $L$  associated with a ball  $B$

- (i)  $r_B \leq \rho_B$
- (ii)  $\text{supp } a \subset B$
- (iii)  $\|a\|_q \leq |B|^{1/q - 1/p}$
- (iv)  $\int a(x) dx = 0$  whenever  $r_B < \rho_B/4$

Let  $\frac{n}{n+\sigma_0} < p \leq 1$  and  $1 < q \leq \infty$ . We then define the atomic Hardy spaces  $H_{L,at}^{p,q}(\mathbb{R}^n)$  as the completion of

$$(3.1) \quad \mathbb{H}_{L,at}^{p,q}(\mathbb{R}^n) = \left\{ f : f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ in } L^2, \ a_j \text{ is an } (p,q)_L\text{-atom and } \sum_j |\lambda_j|^p < \infty \right\}$$

with respect to the norm

$$\|f\|_{H_{L,at}^{p,q}(\mathbb{R}^n)} = \inf \left\{ \left[ \sum_j |\lambda_j|^p \right]^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

We also define the Hardy spaces in terms of finite atoms.

**Definition 3.3.** We define  $H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)$  as the set of all functions  $f = \sum_{j=1}^N \lambda_j a_j$ , where  $a_j$  is an  $(p,q)_L$ -atom if  $q < \infty$  and continuous  $(p,q)_L$ -atom if  $q = \infty$ . For  $f \in H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)$ , we define  $\|f\|_{H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)}$  similarly to  $\|f\|_{H_{L,at}^{p,q}(\mathbb{R}^n)}$ , but the infimum is taken over finite linear decomposition of  $(p,q)_L$ -atoms.

We have the following result.

**Proposition 3.4.** Let  $\frac{n}{n+\sigma_0 \wedge 1} < p \leq 1$  and  $1 < q \leq \infty$ . Then we have the spaces  $H_L^p(\mathbb{R}^n)$  and  $H_{L,at}^{p,q}(\mathbb{R}^n)$  are coincide with equivalent norms.

*Proof.* It was proved in [11] that  $H_L^p(\mathbb{R}^n) \equiv H_{L,at,\infty}^p(\mathbb{R}^n)$ . From definition of  $H_{L,at}^{p,q}(\mathbb{R}^n)$ , we have  $H_{L,at,\infty}^p(\mathbb{R}^n) \hookrightarrow H_{L,at}^{p,q}(\mathbb{R}^n)$ . On the other hand, by a standard argument, see for example [11, 12], we can prove that  $H_{L,at}^{p,q}(\mathbb{R}^n) \hookrightarrow H_L^p(\mathbb{R}^n)$ . This implies that  $H_L^p(\mathbb{R}^n)$  and  $H_{L,at}^{p,q}(\mathbb{R}^n)$  are coincide with equivalent norms.  $\square$

We now prove the following result.

**Proposition 3.5.** Let  $\frac{n}{n+\sigma_0 \wedge 1} < p \leq 1$  and  $1 < q \leq \infty$ . Then the norms  $\|\cdot\|_{H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)}$  and  $\|\cdot\|_{H_{L,at}^{p,q}(\mathbb{R}^n)}$  are equivalent in  $H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)$ .

*Proof.* Let  $f \in H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)$ . Obviously, we have

$$\|f\|_{H_{L,at}^{p,q}(\mathbb{R}^n)} \leq \|f\|_{H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)}.$$

Hence, it suffices to prove the converse inequality. Indeed, we first note that  $f = \sum_{\alpha \in \mathcal{I}_f} \psi_\alpha f$  where  $\mathcal{I}_f = \{\alpha : B_\alpha \cap \text{supp } f \neq \emptyset\}$ . Since  $\text{supp } f$  is bounded, from Lemma 2.2, the set  $\mathcal{I}_f$  is finite. Hence,

$$\|f\|_{H_{L,at,\text{fin}}^{p,q}(M)} \leq \sum_{\alpha \in \mathcal{I}_f} \|\psi_\alpha f\|_{H_{L,at,\text{fin}}^{p,q}(M)}.$$

From the theory of local Hardy spaces in Theorem 3.12 and Theorem 6.2 in [26] (see also [27]), we also get that

$$\sum_{\alpha \in \mathcal{I}_f} \|\psi_\alpha f\|_{H_{L,at,\text{fin}}^{p,q}(\mathbb{R}^n)} \lesssim \sum_{\alpha \in \mathcal{I}_f} \left\| \sup_{0 < t < [\rho(x_\alpha)]^2} |e^{-t\Delta} \psi_\alpha f| \right\|_{L^p(\mathbb{R}^n)}.$$

We now just follows the argument as in [11, p. 53] to conclude that

$$\|f\|_{H_{L,at,\text{fin}}^{p,q}(M)} \lesssim \|\mathcal{M}_L f\|_{L^p(M)}.$$

This completes our proof.  $\square$



**3.2. Molecular characterizations.** In this section we introduce a new kind of molecule, and show that the Hardy spaces  $H_L^p$  can be characterized by such molecules.

**Definition 3.6** (Molecules for  $p = 1$ ). *Let  $1 < q \leq \infty$ . A function  $m$  is called an  $(1, q, \beta)_L$ -molecule for  $H_L^1$  associated to the ball  $B$  if for some  $\beta > 0$*

- (a)  $r_B \leq \rho_B$
- (b)  $\|m\|_{L^q(U_j(B))} \leq 2^{-j\beta} |2^j B|^{1/q-1}$  for all  $j = 0, 1, 2, \dots$
- (c)  $\left| \int_{\mathbb{R}^n} m(x) dx \right| \leq \frac{1}{\log(\rho_B/r_B)}.$

An  $(1, q, \beta)_L$ -molecule associated to the ball  $B$  supported in  $B$  is called an  $(1, q)_{\log}$ -atom.

**Definition 3.7** (Molecules for  $p < 1$ ). *Let  $p \in (0, 1)$  and  $1 < q \leq \infty$ . A function  $m$  is called a  $(p, q, \beta, \delta)_L$ -molecule for  $L$  associated to the ball  $B$  if for some  $\beta, \delta > 0$*

- (a)  $r_B \leq \rho_B$
- (b)  $\|m\|_{L^q(U_j(B))} \leq 2^{-j\beta} |2^j B|^{1/q-1/p}$  for all  $j = 0, 1, 2, \dots$
- (c)  $\left| \int_{\mathbb{R}^n} m(x) dx \right| \leq |B|^{1-1/p} \left( \frac{r_B}{\rho_B} \right)^\delta.$

A  $(p, q, \beta, \delta)_L$ -molecule associated to the ball  $B$  supported in  $B$  is called a  $(p, q, \delta)_L$ -atom.

It is easy to see that a  $(p, q)_L$ -atom is a multiple of a  $(p, q, \beta, \delta)_L$ -molecule for any  $\delta > 0$ ,  $\beta > 0$ . The next result is an almost-orthogonality type estimate for atoms.

**Lemma 3.8.** *Let  $p \in (\frac{n}{n+\sigma_0 \wedge 1}, 1)$ ,  $1 < q \leq \infty$  and  $\delta > 0$ . Let  $a$  be a  $(p, q, \delta)_L$ -atom for  $L$  associated to a ball  $B$  as in Definition 3.7. Then for any  $\nu < \min\{\sigma_0, \delta\}$ , there exists  $C > 0$  so that*

$$|e^{-tL}a(x)| \leq C \frac{r_B^\nu}{|x - x_B|^{n+\nu}} |B|^{1-1/p},$$

for all  $x \in \mathbb{R}^n \setminus 4B$ .

*Proof.* We write

$$e^{-tL}a(x) = \int_B [p_t(x, y) - p_t(x, x_B)] a(y) dy + p_t(x, x_B) \int_B a(y) dy =: I + II$$

Now from the bounds on the heat kernel, and the cancellation for  $a$  we have

$$\begin{aligned} II &\leq |p_t(x, x_B)| \left| \int_B a(y) dy \right| \\ &\lesssim \frac{e^{-|x-x_B|^2/ct}}{t^{n/2}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho_B} \right)^{-N} |B|^{1-1/p} \left( \frac{r_B}{\rho_B} \right)^\nu \\ &\lesssim \frac{t^{\nu/2}}{(\sqrt{t} + |x - x_B|)^{n+\nu}} \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho_B} \right)^{-N} |B|^{1-1/p} \left( \frac{r_B}{\rho_B} \right)^\nu \\ &\lesssim \frac{r_B^\nu}{|x - x_B|^{n+\nu}} |B|^{1-1/p} \end{aligned}$$

by choosing  $N = \nu$ .

Next by using Proposition 2.1 we write

$$\begin{aligned} I &\lesssim \int_B \left( \frac{|y - x_B|}{|x - y|} \right)^\nu \frac{e^{-|x-x_B|^2/ct}}{t^{n/2}} |a(y)| dy \\ &\lesssim \int_B \left( \frac{r_B}{|x - x_B|} \right)^\nu \frac{e^{-|x-x_B|^2/ct}}{t^{n/2}} |a(y)| dy \\ &\lesssim \left( \frac{r_B}{|x - x_B|} \right)^\nu \frac{e^{-|x-x_B|^2/ct}}{t^{n/2}} |B|^{1-1/p} \end{aligned}$$

$$\lesssim \frac{r_B^\nu}{|x - x_B|^{n+\nu}} |B|^{1-1/p}.$$

□

**Lemma 3.9** (Molecules are in  $H_L^p$ ). *Let  $p \in (\frac{n}{n+\sigma_0 \wedge 1}, 1]$  and  $1 < q \leq \infty$ . If  $m$  is a  $(p, q, \beta, \delta)_L$  molecule associated to a ball  $B$  with  $q > 1$  and  $\beta > 0$  and  $\delta > n(1/p - 1)$  then  $m$  is in  $H_L^p$ .*

*Proof.* We divide into two cases:

**Case 1:**  $p < 1$

We wish to show

$$\|\mathcal{M}_L(m)\|_{L^p} \leq C.$$

To do this we set for  $j \geq 0$ ,  $\alpha_j = \int_{U_j(B)} m(x) dx$  and  $\chi_j = \frac{1}{|U_j(B)|} \chi_{U_j(B)}$ . Then we define

$$a_j(x) = m(x) \chi_{U_j(B)}(x) - \alpha_j \chi_j(x).$$

If we set  $N_j = \sum_{k=j}^{\infty} \alpha_k$ , then we have

$$\begin{aligned} m(x) &= \sum_{j=0}^{\infty} a_j(x) + \sum_{j=0}^{\infty} N_{j+1} (\chi_{j+1}(x) - \chi_j(x)) + \chi_0(x) \int m(y) dy \\ (3.2) \quad &= \sum_{j=0}^{\infty} a_j(x) + \sum_{j=0}^{\infty} b_j(x) + a(x), \end{aligned}$$

which implies

$$\begin{aligned} \|\mathcal{M}_L(m)\|_{L^p}^p &\leq \sum_{j=0}^{\infty} \|\mathcal{M}_L(a_j)\|_{L^p}^p + \sum_{j=0}^{\infty} \|\mathcal{M}_L(b_j)\|_{L^p}^p + \|\mathcal{M}_L(a)\|_{L^p}^p \\ &\leq I_1 + I_2 + I_3. \end{aligned}$$

We now take care of the terms in  $I_1$  first. We note that

$$(3.3) \quad \text{supp } a_j \subset 2^j B, \int a_j = 0 \quad \text{and} \quad \|a_j\|_{L^q} \leq C 2^{-j\beta} |2^j B|^{1/q-1/p}.$$

Hence, for  $x \in \mathbb{R}^n \setminus 2^{j+2} B$  we have

$$\begin{aligned} |e^{-tL} a_j(x)| &= \left| \int_{2^j B} [p_t(x, y) - p_t(x, x_B)] a_j(y) dy \right| \\ &\lesssim \int_{2^j B} \left( \frac{|y - x_B|}{|x - y|} \right)^\nu \frac{e^{-|x - x_B|^2/ct}}{t^{n/2}} |a(y)| dy \\ (3.4) \quad &\lesssim \int_{2^j B} \left( \frac{r_B}{|x - x_B|} \right)^\nu \frac{e^{-|x - x_B|^2/ct}}{t^{n/2}} |a(y)| dy \\ &\lesssim 2^{-j\beta} \left( \frac{2^j r_B}{|x - x_B|} \right)^\nu \frac{e^{-|x - x_B|^2/ct}}{t^{n/2}} |2^j B|^{1-1/p} \\ &\lesssim 2^{-j\beta} \frac{(2^j r_B)^\nu}{|x - x_B|^{n+\nu}} |2^j B|^{1-1/p}, \end{aligned}$$

where  $n(1/p - 1) < \nu < \min\{\sigma_0, \delta\}$ .

We now observe that

$$\|\mathcal{M}_L(a_j)\|_{L^p} = \|\mathcal{M}_L(a_j)\|_{L^p(2^{j+2} B)} + \|\mathcal{M}_L(a_j)\|_{L^p(\mathbb{R}^n \setminus 2^{j+2} B)}.$$

Then by the  $L^q$ -boundedness of  $\mathcal{M}$ , Hölder's inequality and (3.3) we have

$$\|\mathcal{M}_L(a_j)\|_{L^p(2^{j+2} B)} \lesssim |2^j B|^{1/p-1/q} \|\mathcal{M}_L(a_j)\|_{L^q(2^{j+2} B)}$$

$$\begin{aligned}
&\lesssim |2^j B|^{1-p/q} \|a_j\|_{L^q} \\
&\lesssim |2^j B|^{1/p-1/q} 2^{-j\beta} |2^j B|^{1/q-1/p} \leq 2^{-j\beta}.
\end{aligned}$$

On the other hand, by (3.4),

$$\|\mathcal{M}_L(a_j)\|_{L^p(\mathbb{R}^n \setminus 2^{j+2}B)} \lesssim 2^{-j\beta} |2^j B|^{1-1/p} \left( \int_{\mathbb{R}^n \setminus 2^{j+2}B} \left( \frac{(2^j r_B)^\nu}{|x - x_B|^{n+\nu}} \right)^p dx \right)^{1/p} \lesssim 2^{-j\beta},$$

as long as  $\nu > n(1/p - 1)$ .

As a consequence,  $I_1 \lesssim \sum_{j \geq 0} 2^{-j\beta} \leq C$ .

Next we also observe that

$$(3.5) \quad \text{supp } b_j \subset 2^{j+1}B \quad \text{and} \quad \int b_j = 0.$$

Moreover,

$$\|b_j\|_{L^q} \leq |N_{j+1}| |2^j B|^{1/q-1}.$$

From the definition of  $N_{j+1}$  and Hölder's inequality, we can get that

$$\begin{aligned}
|N_{j+1}| &\leq \sum_{k \geq j+1} \int_{S_k(B)} |m(y)| dy \leq \sum_{k \geq j+1} |2^k B|^{1-1/q} \|m\|_{L^q(S_k(B))} \\
&\leq \sum_{k \geq j+1} |2^k B|^{1-1/q} 2^{-k\beta} |2^k B|^{1/q-1/p} := \sum_{k \geq j+1} 2^{-k\beta} |2^k B|^{1-1/p} \\
&\leq 2^{-j\beta} \sum_{k \geq j+1} 2^{-(k-j)(\beta+n(1-1/p))} |2^j B|^{1-1/p} \\
&\leq C 2^{-j\beta} |2^j B|^{1-1/p}.
\end{aligned}$$

This implies that

$$(3.6) \quad \|b_j\|_{L^q} \leq C 2^{-j\beta} |2^j B|^{1/q-1/p}.$$

At this stage, an similar argument used to estimate  $I_1$ , we also arrive at that  $I_2 \leq C$ .

For the last term  $I_3$ , we proceed as follows:

$$\|\mathcal{M}_L(a)\|_{L^p} \leq \|\mathcal{M}_L(a)\|_{L^p(4B)} + \|\mathcal{M}_L(a)\|_{L^p(\mathbb{R}^n \setminus 4B)}.$$

For the first term, using the  $L^q$ -boundedness of  $\mathcal{M}_L$  and Hölder' inequality to dominate it by

$$\begin{aligned}
\|\mathcal{M}_L(a)\|_{L^p(4B)} &\leq C |B|^{1/p-1/q} \|a\|_{L^q} \leq C |B|^{1/p-1/q} |B|^{1/q-1} \left| \int m(y) dy \right| \\
&\leq C |B|^{1/p-1} |B|^{1-1/p} \left( \frac{r_B}{\rho_B} \right)^\delta \leq C.
\end{aligned}$$

We now apply Lemma 3.8 to see that

$$\|\mathcal{M}_L(a)\|_{L^p(\mathbb{R}^n \setminus 4B)} \leq C |B|^{1-1/p} \left( \int_{\mathbb{R}^n \setminus 4B} \left[ \frac{r_B^\nu}{|x - x_B|^{n+\nu}} \right]^p \right)^{1/p} \leq C,$$

provided  $\nu > n(1/p - 1)$ .

**Case 2:**  $p = 1$

Similarly to (3.2) we write

$$\begin{aligned}
m(x) &= \sum_{j=0}^{\infty} a_j(x) + \sum_{j=0}^{\infty} N_{j+1}(\chi_{j+1}(x) - \chi_j(x)) + \chi_0(x) \int m(y) dy \\
&= \sum_{j=0}^{\infty} a_j(x) + \sum_{j=0}^{\infty} b_j(x) + a(x).
\end{aligned}$$

The argument as in Case 1 has shown that  $\sum_{j=0}^{\infty} a_j(x) + \sum_{j=0}^{\infty} b_j(x)$  is in  $H_L^1$ . It remains to show that  $a(x) := \chi_0(x) \int m(y) dy \in H_L^1$ . By Proposition 3.12 we claim that

$$\left| \int_B a(x) \phi(x) dx \right| \leq C \|\phi\|_{BMO_L},$$

for all  $\phi \in C^\infty(\mathbb{R}^n)$ .

Indeed, we have

$$\left| \int_B a(x) \phi(x) dx \right| \leq \left| \int_B a(x) (\phi(x) - \phi_B) dx \right| + |\phi_B| \left| \int_B a(x) dx \right|.$$

By Hölder's inequality we have

$$\begin{aligned} \left| \int_B a(x) (\phi(x) - \phi_B) dx \right| &\leq \|a\|_{L^q(B)} \left( \int_B |\phi(x) - \phi_B|^{q'} dx \right)^{1/q'} \\ &\leq C |B|^{1/q-1} |B|^{1/q'} \|\phi\|_{BMO_L} := C \|\phi\|_{BMO_L}. \end{aligned}$$

To dominate the second term we note that by [9, Lemma 2], we have

$$|\phi_B| \lesssim \|\phi\|_{BMO_L} \log \left( \frac{\rho_B}{r_B} \right).$$

Inserting this into the second term to obtain that

$$|\phi_B| \left| \int_B a(x) dx \right| \lesssim \|\phi\|_{BMO_L} \log \left( \frac{\rho_B}{r_B} \right) \left| \int_B m(x) dx \right| \lesssim \|\phi\|_{BMO_L}.$$

This completes our proof.  $\square$

**Proposition 3.10** (Molecular characterization).  $H_L^1(\mathbb{R}^n)$  is equivalent to the completion of (3.7)

$$\mathbb{H}_{L,mol}^{1,q}(\mathbb{R}^n) = \{f : f = \sum_{j=1}^{\infty} \lambda_j m_j \text{ in } L^2, \ m_j \text{ is an } (1, q, \beta)_L\text{-molecule and } \sum_j |\lambda_j|^p < \infty\}$$

with respect to the norm

$$\|f\|_{\mathbb{H}_{L,mol}^{1,q}(\mathbb{R}^n)} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j m_j \right\}.$$

For  $p \in (\frac{n}{n+\sigma_0}, 1)$  then  $H_L^p$  is equivalent to the completion of (3.8)

$$\mathbb{H}_{L,mol}^{p,q}(\mathbb{R}^n) = \{f : f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ in } L^2, \ m_j \text{ is an } (p, q, \beta, \delta)_L\text{-atom and } \sum_j |\lambda_j|^p < \infty\}$$

with respect to the norm

$$\|f\|_{\mathbb{H}_{L,mol}^{p,q}(\mathbb{R}^n)} = \inf \left\{ \left[ \sum_j |\lambda_j|^p \right]^{1/p} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

*Proof.* Combining the Lemmas 3.8 and 3.9 together with the atomic characterization of  $H_L^p(\mathbb{R}^n)$ , we obtain this proposition.  $\square$

**3.3. Campanato spaces.** We now recall the definition of Campanato spaces associated to the Schrödinger operators.

**Definition 3.11.** Let  $\alpha \in [0, 1)$ . We set

$$BMO_L^\alpha = \{f \in L_{loc}^1 : \|f\|_{BMO_L^\alpha} < \infty\}$$

where  $\|f\|_{BMO_L^\alpha}$  is the infimum of all  $C > 0$  such that

$$\frac{1}{|B|^{1+\alpha/n}} \int_B |f - f_B| \leq C$$

for all balls  $B$ , and

$$\frac{1}{|B|^{1+\alpha/n}} \int_B |f| \leq C$$

for all balls  $B$  with  $r_B \geq \rho_B$ .

Note that in the particular case when  $\alpha = 0$ , the Campanato space  $BMO_L^\alpha$  turns out to be the BMO space  $BMO_L$  which introduced in [9]. For the general case when  $\alpha \in (0, 1)$ , these spaces were first introduced in [3] to consider the boundedness of generalized fractional integrals  $L^{-\gamma/2}$ ,  $\gamma > 0$  related to Schrödinger operators whose potentials satisfy certain reverse Hölder inequality. Recently, the theory of generalized Morrey-Campanato spaces associated to admissible functions has been investigated in [30, 31]. These spaces include the Campanato type spaces in various settings of Schrödinger operators such as Schrödinger operators, degenerate Schrödinger operators on  $\mathbb{R}^n$  and Schrödinger operators on Heisenberg groups and connected and simply connected nilpotent Lie groups.

It is clear from their definitions that  $BMO_L^\alpha \subset BMO_L$  and that for  $\alpha = 0$  we have  $BMO_L^\alpha = BMO_L$ . Furthermore for  $\alpha > 1$ , the spaces  $BMO_L^\alpha$  contain only constant functions. They also coincide with the space of Lipschitz continuous functions. Indeed if we define  $\Lambda_L^\alpha$  to be the space of continuous functions  $f$  for which

$$\|f\|_{\Lambda_L^\alpha} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} + \sup_{x \in \mathbb{R}^n} |\rho(x)^{-\alpha} f(x)|$$

is finite, then  $BMO_L^\alpha$  and  $\Lambda_L^\alpha$  coincide for all  $0 < \alpha \leq 1$  with equivalent norms. See for example [4, 30, 31].

It is important to note that the Campanato spaces are the duals of the Hardy spaces. In fact, in the case  $p = 1$ , it was proved in [9] that  $(H_L^1)^* = BMO_L$ . For  $p \in (\frac{n}{n+1}, 1)$ , we have

$$(3.9) \quad (H_L^p)^* = BMO_L^{n(\frac{1}{p}-1)}.$$

See for example [30]. For the predual space of the Hardy spaces  $H_L^1$  we have the following result in [19, Theorem 4.1].

**Proposition 3.12.** Let  $CMO_L$  be the closure of  $C^\infty(\mathbb{R}^n)$  in  $BMO_L$ . Then,  $H_L^1$  is the dual space of  $CMO_L$ .

We will summarize some properties involving the  $BMO_L^\alpha$  spaces.

**Proposition 3.13.** Let  $\alpha \geq 0$  and  $p \in [1, \infty)$ . Then the following statement holds:

(i) A function  $f$  belongs to the  $BMO_L^\alpha$  space if and only if

$$(3.10) \quad \sup_{B: \text{ball}} \left( \frac{1}{|B|^{1+p\alpha/n}} \int_B |f(x) - f_B|^p dx \right)^{1/p} + \sup_{B: r_B \geq \rho_B} \left( \frac{1}{|B|^{1+p\alpha/n}} \int_B |f(x)|^p dx \right)^{1/p} < \infty.$$

Moreover, the left hand side of (3.10) is comparable with  $\|f\|_{BMO_L^\alpha}$ .

(ii) For all balls  $B := B(x_0, r)$  with  $r < \rho(x_0)$  and  $f \in BMO_L^\alpha$ , we have

$$\frac{1}{|B|^{1+\alpha/n}} \int_B |f(x)| dx \lesssim \begin{cases} \left(\frac{\rho(x_0)}{r}\right)^\alpha \|f\|_{BMO_L^\alpha}, & \alpha > 0 \\ \left[1 + \log\left(\frac{\rho(x_0)}{r}\right)\right] \|f\|_{BMO_L^\alpha}, & \alpha = 0. \end{cases}$$

(iii) For all  $x \in \mathbb{R}^n$  and  $0 < r_1 < r_2$ ,

$$|f_{B(x, r_1)} - f_{B(x, r_2)}| \lesssim \begin{cases} \left(\frac{r_2}{r_1}\right)^\alpha |B(x, r_1)|^{\alpha/n} \|f\|_{BMO_L^\alpha}, & \alpha > 0 \\ \left[1 + \log\left(\frac{r_2}{r_1}\right)\right] \|f\|_{BMO_L}, & \alpha = 0. \end{cases}$$

*Proof.* For the proof, we refer the reader to Lemma 2.2 and Lemma 2.4 in [30].  $\square$

#### 4. PROOF OF THE $T1$ CRITERIONS FOR $H_L^p(\mathbb{R}^n)$ AND $BMO_L^\alpha(\mathbb{R}^n)$

Before coming to the proof of the main result, we would like to give the definition of  $T^*f$  for  $f \in BMO_L^\alpha$ ,  $0 < \alpha \leq 1$  and  $T \in GCZO(\gamma, \theta)$ . Let  $K^*(x, y)$  be an associated kernel of  $T^*$ . Following the ideas in [22], we can define  $T^*f$  for  $f \in BMO_L^\alpha$ ,  $0 < \alpha \leq 1$ . For the sake of convenience, we just sketch it here.

Fix  $x_0 \in \mathcal{R}^n$ . For  $R > \rho(x_0)$  we define

$$T^*f(x) = T^*(f\chi_{B(x_0, R)})(x) + \int_{B(x_0, R)^c} K^*(x, y) f(y) dy.$$

Since  $f\chi_{B(x_0, R)} \in L_c^{\theta'}$  and  $T^*$  is bounded on  $L^{\theta'}$ , the first term is well-defined.

For the second term, using (1.5), Proposition 3.13 and Lemma 2.2 (i) we can dominate the second term by

$$CR^\alpha \|f\|_{BMO_L^\alpha}.$$

Similarly to [22], we can show that  $T^*f$  is independent of  $R$  in the sense that if  $B(x_0, R) \subset B(x'_0, R')$  then the definition using  $B(x'_0, R')$  coincides with the one using  $B(x_0, R)$  for a.e.  $B(x_0, R)$ .

Since  $1 \in BMO_L^\alpha$ , the definition above is valid for  $T^*1$ .

Now for  $f \in BMO_L^\alpha$ ,  $0 < \alpha \leq 1$ . For any ball  $B$  we have

$$f = (f - f_B)\chi_{4B} + (f - f_B)\chi_{(4B)^c} + f_B := f_1 + f_2 + f_3.$$

Arguing similarly to [22], we also obtain that

$$T^*f = T^*f_1 + T^*f_2 + T^*f_3.$$

We are now ready to give the proof of Theorem 1.2.

#### **Proof of Theorem 1.2.** Proof of “if part” for (a) and (b)

For  $p \in (0, 1]$  with  $\gamma > n(1/p - 1)$ , it suffices to show that  $T$  maps  $(p, \theta)_L$ -atoms into molecules  $(p, \theta, \epsilon)_L$ -molecules as  $p = 1$  and into  $(p, \theta, \epsilon, \delta)_L$ -molecules as  $p < 1$  with  $0 < \epsilon < \gamma - n(1/p - 1)$  and  $\delta = n(1/p - 1)$ .

Indeed, let  $a$  be an  $(p, \theta)_L$ -atom associated to a ball  $B$ . We first prove the size condition on  $Ta$ . If  $j = 0, 1, 2, 3$  then  $L^\theta$ -boundedness of  $T$  implies that

$$\|Ta\|_{L^\theta(4B)} \lesssim \|a\|_{L^\theta} \lesssim |B|^{1/\theta - 1/p}.$$

For  $j \geq 4$  we consider two cases:

*Case 1:*  $r_B < \rho_B/4$ .

In this situation by using the cancelation property, Minkowski's inequality and (1.6) we can write

$$\|Ta\|_{L^\theta(U_j(B))} = \left( \int_{U_j(B)} \left( \int |K(x, y) - K(x, x_B)| |a(y)| dy \right)^\theta dx \right)^{1/\theta}$$

$$\begin{aligned}
&\leq \int_B \left( \int_{U_j(B)} |K(x, y) - K(x, x_B)|^\theta dx \right)^{1/\theta} |a(y)| dy \\
&\lesssim 2^{-j\gamma} |2^j B|^{-1/\theta'} \|a\|_{L^1} \\
&\lesssim 2^{-j\gamma} |2^j B|^{-1/\theta'} |B|^{1-1/p} = 2^{-j[\gamma-n(1-1/p)]} |2^j B|^{-1/p\theta'} |2^j B|^{1/\theta-1/p}.
\end{aligned}$$

Case 2:  $\rho_B/4 \leq r_B \leq \rho_B$ .

In this situation, by Minkowski's inequality we write

$$\|Ta\|_{L^\theta(U_j(B))} = \left( \int_{U_j(B)} \left| \int_B K(x, y) a(y) dy \right|^\theta dx \right)^{1/\theta} \leq \int_B \left( \int_{U_j(B)} |K(x, y)|^\theta dx \right)^{1/\theta} |a(y)| dy.$$

This along with (1.5) yields that, for  $N > \gamma$ ,

$$\|Ta\|_{L^\theta(U_j(B))} \lesssim |2^j B|^{-1/\theta'} \left( \frac{\rho_B}{2^j r_B} \right)^N \|a\|_{L^1} \lesssim 2^{-j[\gamma-n(1/p-1)]} |2^j B|^{1/\theta-1/p}.$$

To obtain the cancellation for  $Ta$ , we make the following split

$$\begin{aligned}
\left| \int Ta(x) dx \right| &= \left| \int a(x) T^* 1(x) dx \right| \\
&\leq \int |a(x)| |T^* 1(x) - (T^* 1)_B| dx + \left| \int a(x) dx \right| |(T^* 1)_B| \\
&=: I + II
\end{aligned}$$

To estimate the first term, we may apply Hölder's inequality to obtain, for  $p < 1$ ,

$$\begin{aligned}
I &\leq \|a\|_\theta \|T^* 1 - (T^* 1)_B\|_{L^{\theta'}(B)} \\
&\leq |B|^{1-1/p} \left( \int_B |T^* 1(x) - (T^* 1)_B|^{\theta'} dx \right)^{1/\theta'} \\
&\lesssim |B|^{1-1/p} \left( \frac{r_B}{\rho_B} \right)^{n(1/p-1)}
\end{aligned}$$

If  $p = 1$  then

$$I \leq \left( \int_B |T^* 1(x) - (T^* 1)_B|^{\theta'} dx \right)^{1/\theta'} \lesssim \frac{1}{\log \left( \frac{\rho_B}{r_B} \right)}$$

To estimate the second term, we note that if  $r_B \leq \rho_B/4$  then  $\int a = 0$  and hence  $II = 0$ . Otherwise we have  $\rho_B/4 \leq \frac{r_B}{\rho_B} \leq \rho_B$  and therefore

$$II \leq |(T^* 1)_B| \left| \int a(x) dx \right| \lesssim |B|^{1-1/p} \lesssim |B|^{1-1/p} \left( \frac{r_B}{\rho_B} \right)^{n(1/p-1)}$$

If  $p = 1$  then we argue similarly but use

$$II \lesssim \left( \frac{r_B}{\rho_B} \right)^\delta \lesssim \frac{1}{\log \left( \frac{\rho_B}{r_B} \right)}$$

for any  $\delta > 0$ .

Proof of 'only if' of (a)

We borrow some ideas in [22]. Assume that  $T$  is bounded on  $H_L^1$  then from (3.9)  $T^*$  is bounded on  $BMO_L$ . For  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \rho(x_0)$  we define

$$g_{x_0, s}(x) = \chi_{[0, s]}(|x - x_0|) \log \left( \frac{\rho(x_0)}{s} \right) + \chi_{(s, \rho(x_0)]}(|x - x_0|) \log \left( \frac{\rho(x_0)}{|x - x_0|} \right).$$

Then we have  $g_{x_0, s} \geq 0$  and  $\|g_{x_0, s}\|_{BMO_L} \leq C$ . See [22, Lemma 2.5].

We now fix  $x_0 \in \mathbb{R}^n$  and  $0 < s \leq \rho(x_0)/2$ . Set  $B = B(x_0, s)$  and  $g_0(x) = g_{x_0, s}(x)$ .

We split  $f_0 = (f_0 - (f_0)_B)\chi_{4B} + (f_0 - (f_0)_B)\chi_{(4B)^c} + (f_0)_B := f_1 + f_2 + (f_0)_B$  which implies that

$$(f_0)_B T^* 1(y) = T^* f_0(y) - T^* f_1(y) - T^* f_2(y).$$

Therefore,

$$\begin{aligned} (f_0)_B \log\left(\frac{\rho(x_0)}{s}\right) \left( \int_B |T^* 1(y) - (T^* 1)_B|^{\theta'} dy \right)^{1/\theta'} \\ \leq \sum_{i=0,1,2} \log\left(\frac{\rho(x_0)}{s}\right) \left( \int_B |T^* f_i(y) - (T^* f_i)_B|^{\theta'} dy \right)^{1/\theta'} \\ := I_0 + I_1 + I_2. \end{aligned}$$

From the  $BMO_L$ -boundedness of  $T^*$  and Proposition 3.13 we obtain

$$\begin{aligned} I_0 &\lesssim \log\left(\frac{\rho(x_0)}{s}\right) \|T^* f_0\|_{BMO_L} \lesssim \log\left(\frac{\rho(x_0)}{s}\right) \|f_0\|_{BMO_L} \\ &\lesssim \log\left(\frac{\rho(x_0)}{s}\right) := (f_0)_B. \end{aligned}$$

For the contribution of  $I_1$ , we have

$$\begin{aligned} \left( \int_B |T^* f_1(y) - (T^* f_1)_B|^{\theta'} dy \right)^{1/\theta'} &\leq 2 \left( \int_B |T^* f_1(y)|^{\theta'} dy \right)^{1/\theta'} \\ &\lesssim |B|^{-1/\theta'} \|T^* f_1\|_{L^{\theta'}}. \end{aligned}$$

This in combination with the  $L^{\theta'}$ -boundedness of  $T^*$  and Proposition 3.13 implies that

$$\left( \int_B |T^* f_1(y) - (T^* f_1)_B|^{\theta'} dy \right)^{1/\theta'} \lesssim \|f_0\|_{BMO_L}.$$

Hence,  $I_1 \lesssim (f_0)_B$ .

For the last term  $I_2$ , using Hölder's inequality and (1.6) we have for  $y \in B$

$$\begin{aligned} &|T^* f_2(y) - (T^* f_2)_B| \\ &\leq \frac{1}{|B|} \int_B \int_{(4B)^c} |K(z, y) - K(z, u)| |f_0(z) - (f_0)_B| dz du \\ &\leq \sum_{k \geq 1} \frac{1}{|B|} \int_B \left( \int_{S_k(B)} |K(z, y) - K(z, u)|^\theta dz \right)^{1/\theta} \left( \int_{S_k(B)} |f_0(z) - (f_0)_B|^{\theta'} dz \right)^{1/\theta'} du \\ &\leq \sum_{k \geq 1} 2^{-k\gamma} |2^k B|^{-1/\theta'} \left( \int_{S_k(B)} |f_0(z) - (f_0)_B|^{\theta'} dz \right)^{1/\theta'} \\ &\leq \sum_{k \geq 1} 2^{-k\gamma} \left[ \left( \frac{1}{|2^k B|} \int_{2^k B} |f_0(z) - (f_0)_B|^{\theta'} dz \right)^{1/\theta'} + |(f_0)_{2^k B} - (f_0)_B| \right] \end{aligned}$$

which along with Proposition 3.13 yields that

$$|T^* f_2(y) - (T^* f_2)_B| \leq \sum_{k \geq 1} 2^{-k\gamma} \log(2^k) \|f_0\|_{BMO_L} \leq C.$$

Hence,  $I_2 \lesssim (f_0)_B$ .

Taking the estimates of  $I_0, I_1$  and  $I_2$  into account implies that

$$\log\left(\frac{\rho(x_0)}{s}\right) \left( \int_B |T^* 1(y) - (T^* 1)_B|^{\theta'} dy \right)^{1/\theta'} \leq C.$$

This completes our proof.



Proof of ‘only if’ of (b)

Assume that  $T$  is bounded on  $H_L^p(\mathbb{R}^n)$ . We point out that in Section 3.1 of [22], they provided a definition of  $T^*f(x)$  for a.e.  $x \in B(x_0, R)$ , for  $f \in BMO_L^{n(1/p-1)}$ ,  $R \geq \rho(x_0)$  and  $x_0 \in \mathbb{R}^n$ . Hence, by Proposition 3.5, suppose  $g = \sum_{j=1}^n a_j \in H_L^p(\mathbb{R}^n)$ , where each  $a_j$  is an  $(p, q)_L$ -atom if  $q < \infty$  and continuous  $(p, q)_L$ -atom if  $q = \infty$ . Then we obtain that for every  $f \in BMO_L^{n(1/p-1)}$ ,

$$\langle T^*f, g \rangle = \langle f, Tg \rangle \lesssim \|f\|_{BMO_L^{n(1/p-1)}} \|Tg\|_{H_L^p(\mathbb{R}^n)} \lesssim \|f\|_{BMO_L^{n(1/p-1)}} \|g\|_{H_L^p(\mathbb{R}^n)}.$$

Taking the supremum over all  $g$  gives

$$\|T^*f\|_{BMO_L^{n(1/p-1)}} \lesssim \|f\|_{BMO_L^{n(1/p-1)}}.$$

This implies that  $T^*$  is bounded on  $BMO_L^\alpha$  with  $\alpha = n(1/p - 1)$ .

For  $x_0 \in \mathbb{R}^n$  and  $0 < s < \rho(x_0)$ , we define

$$g_{x_0, s}(x) = \chi_{[0, s]}(|x - x_0|)(\rho(x_0)^\alpha - s^\alpha) + \chi_{(s, \rho(x_0)]}(|x - x_0|)(\rho(x_0)^\alpha - s^\alpha).$$

We then have  $g \geq 0$  and  $\|g_{x_0, s}\|_{BMO_L^\alpha} \leq C$ . See [22, Lemma 2.5].

The remainder of the proof is similar to that of the “only if” part for (a). The difference here is that we must apply for the function  $g_{x_0, s}$  instead of  $f_0$ . Hence, we omit details here.

**Proof of Theorem 1.2 (c)**Proof of ‘if’

We first recall the notion of the classical  $(p, q, \epsilon)$ -molecule for  $H^p$  with  $\frac{n}{n+1} < p \leq 1$ . For  $\frac{n}{n+1} < p \leq 1 \leq q < \infty$  and  $\epsilon > 0$ , a function  $m$  is said to be a  $(p, q, \epsilon)$ -molecule if there holds

- (i)  $\|m\|_{L^q(U_j(B))} \leq 2^{-j\epsilon} |2^j B|^{1/q-1/p}$  for all  $j \geq 0$
- (ii)  $\int m(x) dx = 0$ .

It is well-known that if  $m$  is a  $(p, q, \epsilon)$ -molecule then  $\|m\|_{H^p} \leq C$ . Hence, to prove this part it suffices to prove that  $T$  maps each  $(p, \theta)_L$  atom into  $(p, \theta, \epsilon)$  molecule for some  $\epsilon > 0$ .

Indeed, let  $a$  be a  $(p, \theta)_L$  atom associated to  $B$ . We consider two cases:  $r_B < \rho_B/4$  and  $\rho_B/4 \leq r_B \leq \rho_B$ . The first case is very standard. Hence, we need to consider the second case  $\rho_B/4 \leq r_B \leq \rho_B$ .

We first observe that from the condition  $T^*1 = 0$  we have  $\int Ta(x) dx = 0$ . To complete the proof, we need only to show that

$$(4.1) \quad \|Ta\|_{L^\theta(U_j(B))} \lesssim 2^{-j\epsilon} |2^j B|^{1/\theta-1/p}, \quad j \geq 0.$$

From the  $L^\theta$ -boundedness of  $T$  it can be verified that (4.1) holds true for  $j = 0, 1, 2$ .

Fix  $N > n(1/p - 1)$ . For  $j \geq 3$  by (1.5) and Minkowski's inequality we have

$$\begin{aligned} \|Ta\|_{L^\theta(U_j(B))} &\leq \left[ \int_{U_j(B)} \left| \int_B |K(x, y) a(y) dy|^\theta dx \right]^{1/\theta} \\ &\leq \int_B \left[ \int_{U_j(B)} |K(x, y)|^\theta dx \right]^{1/\theta} |a(y)| dy \\ &\lesssim |2^j B|^{-1/\theta'} 2^{-jN} \|a\|_1 \\ &\lesssim |2^j B|^{-1/\theta'} 2^{-jN} |B|^{1-1/p} := 2^{-j[N-n(1/p-1)]} |2^j B|^{1/\theta-1/p}. \end{aligned}$$

This proves (4.1).

Proof of ‘only if’ Assume that  $T$  is bounded from  $H_L^p$  into  $H^p$ . Then by duality,  $T^*$  maps  $BMO^\alpha$  into  $BMO_L^\alpha$  continuously with  $\alpha = n(1/p - 1)$ . Then we have

$$\|T^*1\|_{BMO_L^\alpha} \leq C\|1\|_{BMO^\alpha} = 0.$$

Hence,  $\|T^*1\|_{BMO_L^\alpha} = 0$ . From the definition of  $BMO_L^\alpha$  we have  $\int_B |T^*1| = 0$  for all  $B = B(x, \rho(x))$  with  $x \in \mathbb{R}^n$ . This along with Lemma 2.4 implies  $\int_{\mathbb{R}^n} |T^*1| = 0$ . It follows that  $T^*1 = 0$ .  $\square$

**Proof of Theorem 1.4.** Since  $T \in GCZO_\rho^*(\gamma, \theta')$  implies that  $T^* \in GCZO_\rho(\gamma, \theta)$ , the proof of the ‘if’ directions follow from Theorem 1.2 and duality.

We also observe that the proofs of the ‘only if’ directions are essentially contained in the proofs of the ‘only if’ directions in Theorem 1.2.  $\square$

## 5. PROOFS OF APPLICATIONS

In this section we give the proofs of Theorems 1.5–1.8.

**5.1. Laplace transform type multipliers.** Suppose  $L$  is the Schrödinger operator defined as in (1.1). Given a bounded function  $a : [0, \infty) \rightarrow \mathbb{C}$ , we define the *Laplace transform type multipliers*  $m(L)$  by

$$(5.1) \quad m(L)f(x) = \int_0^\infty a(t)Le^{-tL}f(x)dt$$

which is bounded on  $L^2$ . An example are the imaginary powers  $m(L) = L^{i\nu}$  given by  $a(t) = -\frac{1}{\Gamma(i\nu)}t^{-i\nu}$  for  $\nu \in \mathbb{R}$ .

**Proof of Theorem 1.5.** We now apply Theorem 1.2 to prove Theorem 1.5.

Denote by  $m(L)(x, y)$  the associated kernel of  $m(L)$ . Then it was proved in [22] that

**Proposition 5.1.** *Let  $x, y, z \in \mathbb{R}^n$  and  $N > 0$ . Then*

- (a)  $|m(L)(x, y)| \leq \frac{C}{|x - y|^n} \left(1 + \frac{|x - y|}{\rho(x)} + \frac{|x - y|}{\rho(y)}\right)^{-N};$
- (b)  $|m(L)(x, y) - m(L)(x, z)| + |m(L)(y, x) - m(L)(z, x)| \leq C_\delta \frac{|y - z|^\delta}{|x - y|^{n+\delta}},$  for all  $|x - y| > 2|y - z|$  and any  $0 < \delta < \sigma_0$ .

Fix  $\frac{n}{n+\sigma_0 \wedge 1} < p \leq 1$  and take  $\delta < \sigma_0 \wedge 1$  so that  $\frac{n}{n+\delta} < p \leq 1$ . From Proposition 5.1,  $m(L) \in GCZO(\delta, 2)$ . Hence, in the light of Theorem 1.2 and the fact that  $m(L)^* = \overline{m}(L)$  it suffices to prove that

$$(5.2) \quad \log\left(\frac{\rho_B}{r_B}\right) \left( \int_B |m(L)1(y) - (m(L)1)_B|^2 dy \right)^{1/2} \leq C,$$

$$(5.3) \quad \left(\frac{\rho_B}{r_B}\right)^{n(1/p-1)} \left( \int_B |m(L)1(y) - (m(L)1)_B|^2 dy \right)^{1/2} \leq C$$

for every ball  $B$  with  $r_B \leq \frac{1}{2}\rho_B$ .

Indeed, we have by Minkowski's inequality

$$\begin{aligned} \left( \int_B |m(L)1(y) - (m(L)1)_B|^2 dy \right)^{1/2} &\lesssim \left( \int_B \left| \int_B m(L)1(y) - m(L)1(z) dz \right|^2 dy \right)^{1/2} \\ &\lesssim \int_B \left( \int_B |m(L)1(y) - m(L)1(z)|^2 dy \right)^{1/2} dz. \end{aligned}$$

It was proved in the proof of [22, Proposition 4.11] that

$$|m(L)1(y) - m(L)1(z)| \lesssim \left(\frac{r_B}{\rho_B}\right)^\delta \log\left(\frac{\rho_B}{r_B}\right).$$

Hence,

$$\left(\int_B |m(L)1(y) - (m(L)1)_B|^2 dy\right)^{1/2} \lesssim \left(\frac{r_B}{\rho_B}\right)^\delta \log\left(\frac{\rho_B}{r_B}\right).$$

This proves (5.2) and (5.3).  $\square$

**5.2. Riesz transforms  $\nabla L^{-1/2}$  and  $\nabla^2 L^{-1}$ .** Suppose  $L$  is the Schrödinger operator defined as in (1.1). For  $i, j = 1, \dots, n$ , the  $i$ -th Riesz transform is defined by

$$\mathcal{R}_i = \partial_{x_i} L^{-1/2} = \frac{1}{\pi} \int_0^\infty \partial_{x_i} e^{-tL} \frac{dt}{\sqrt{t}},$$

and the  $i, j$ -th Riesz transform is defined by

$$\mathcal{R}_{ij} = \partial_{x_i} \partial_{x_j} L^{-1} = \int_0^\infty \partial_{x_i} \partial_{x_j} e^{-tL} dt.$$

For simplicity we shall write  $\nabla$  and  $\nabla^2$  for  $\partial_{x_i}$  and  $\partial_{x_i} \partial_{x_j}$  respectively, and set  $\mathcal{R}_{(1)} := \nabla L^{-1/2}$  and  $\mathcal{R}_{(2)} := \nabla^2 L^{-1}$ .

**Proof of Theorem 1.6.** We first consider  $\mathcal{R}_{(1)}$ . Now  $\mathcal{R}_{(1)} \in GCZO(\delta, 2)$  for any  $0 < \delta < \min\{\sigma_0, 1\}$ . Indeed, it is well-known that  $\mathcal{R}_{(1)}$  is bounded on  $L^2$ . The condition (1.5) and (1.6) follow from [4, Lemma 7] and [14], respectively. On the other hand, it is obvious  $\mathcal{R}_{(1)}^* 1 = 0$ . The conclusion of the theorem follows immediately by applying Theorem 1.2 (c).

We now consider  $\mathcal{R}_{(2)}$ .

We will show that  $\mathcal{R}_{(2)} \in GCZO_\rho(\delta, \sigma)$  for any  $0 < \delta < \min\{\sigma_0, 1\}$ . Then observing that  $\mathcal{R}_{(2)}^* 1 = 0$ , the conclusion of the theorem follows from Theorem 1.2 (c) also. The boundedness of  $\mathcal{R}_{(2)}$  on  $L^\sigma(\mathbb{R}^n)$  for  $n \geq 3$  was established in [23]. It remains to prove (1.5) and (1.6). The following kernel estimates are required.

**Proposition 5.2.** *For each  $1 \leq \theta \leq \sigma$ , there exists  $\kappa > 0$  such that the following holds for all  $N > 0$ .*

(a) *For every  $y \in \mathbb{R}^n$ ,  $t > 0$ ,*

$$\left\| \nabla^2 p_t(\cdot, y) e^{\frac{|\cdot - y|^2}{\kappa t}} \right\|_{L^\theta} \leq C t^{-1 - \frac{n}{2\theta'}} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

(b) *For all  $|y - y'| \leq \sqrt{t}$  and any  $0 < \sigma_1 < \sigma_0$  we have*

$$\left\| [\nabla^2 p_t(\cdot, y) - \nabla^2 p_t(\cdot, y')] e^{\frac{|\cdot - y|^2}{\kappa t}} \right\|_{L^\theta} \leq C \left(\frac{|y - y'|}{\sqrt{t}}\right)^{\sigma_1} t^{-1 - \frac{n}{2\theta'}} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

*Proof of Proposition 5.2.* Part (a) was proved in [21] Proposition 2.4. Part (b) can be obtained by the same argument but using the second estimate of Proposition 2.1 in place of the first.  $\square$

We these estimates in hand, we can now obtain (1.5) and (1.6) for the kernel of  $\mathcal{R}_{(2)}$  given by

$$K(x, y) = \int_0^\infty \nabla_x^2 p_t(x, y) dt.$$

In fact the proof of (1.5) and (1.6) is the same as that of  $K_s(x, y)$  for the operator  $V^s L^{-s}$  for  $s = 1$  (see (5.5)–(5.7) below), but applying Proposition 5.2 in place of Proposition 5.3.  $\square$

**5.3. Riesz transforms  $V^s L^{-s}$ ,  $0 < s \leq 1$ , and their adjoints.** For each  $0 < s \leq 1$  we set

$$V^s L^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty V^s e^{-tL} \frac{dt}{t^{1-s}}.$$

It is known from Corollary 3 of [25] that the operators  $V^s L^{-s}$  are bounded on  $L^p(\mathbb{R}^n)$  for each  $1 < p < \frac{\sigma}{s}$ .

**Proof of Theorem 1.7.** To prove this theorem we shall apply Theorem 1.2 to  $T = V^s L^{-s}$ . We first show that  $V^s L^{-s} \in GCZO_\rho(\gamma, \theta)$  for any  $1 < \theta < \sigma/s$  and  $0 < \gamma < \sigma_0$ . To do so, we require the following kernel estimates for  $V^s e^{-tL}$ .

**Proposition 5.3.** *For each  $0 < s \leq 1$  and  $1 \leq \theta \leq \frac{\sigma}{s}$ , there exists  $\kappa > 0$  such that the following holds for all  $N > 0$ .*

(a) *For every  $y \in \mathbb{R}^n$ ,  $t > 0$ ,*

$$\left\| V^s(\cdot) p_t(\cdot, y) e^{\frac{|\cdot-y|^2}{\kappa t}} \right\|_{L^\theta} \leq C t^{-s-\frac{n}{2\theta'}} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}.$$

(b) *For all  $|y - y'| \leq \sqrt{t}$  and any  $0 < \sigma_1 < \sigma_0$  we have*

$$\left\| V^s(\cdot) [p_t(\cdot, y) - p_t(\cdot, y')] e^{\frac{|\cdot-y|^2}{\kappa t}} \right\|_{L^\theta} \leq C \left( \frac{|y - y'|}{\sqrt{t}} \right)^{\sigma_1} t^{-s-\frac{n}{2\theta'}}$$

*Proof of Proposition 5.3.* We need the following estimate: for  $N$  large enough we have

$$(5.4) \quad \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left( t \int_{B(x, \sqrt{t})} V \right)^q \leq C_{N,q}.$$

We can see this by applying Remark 2.3.

For the proof of (a), by applying the bounds on the heat kernel  $p_t(x, y)$  from Proposition 2.1 and by taking  $\kappa$  large enough we have

$$\left\| V^s(\cdot) p_t(\cdot, y) e^{\frac{|\cdot-y|^2}{\kappa t}} \right\|_{L^\theta}^\theta \lesssim t^{-\frac{n\theta}{2}} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N'\theta} \int V(x)^{s\theta} e^{-c\frac{|x-y|^2}{t}} dx.$$

Now since  $V^s \in RH_{\sigma/s}$  and  $\theta \leq \sigma/s$  then

$$\begin{aligned} \int V(x)^{s\theta} e^{-c\frac{|x-y|^2}{t}} dx &= \int_{B(y, 2\sqrt{t})} \dots dx + \sum_{j=1}^\infty \int_{B(y, 2^{j+1}\sqrt{t}) \setminus B(y, 2^j\sqrt{t})} \dots dx \\ &\lesssim t^{n/2} \left( \int_{B(y, \sqrt{t})} V^s \right)^\theta \left\{ 1 + \sum_{j=1}^\infty e^{-c4^j} 2^{j(n_{0,s}+n-n\theta)} \right\} \\ &\lesssim t^{n/2-s\theta} \left( \int_{B(y, \sqrt{t})} V \right)^{s\theta}, \end{aligned}$$

where in the second last step we applied the doubling property of  $V^s$ , with  $n_{0,s}$  the doubling power of  $V^s$ . In the last step we applied Hölder's inequality with exponent  $1/s$ .

Therefore in view of (5.4) and by choosing  $N'$  large enough we obtain

$$\begin{aligned} \left\| V^s(\cdot) p_t(\cdot, y) e^{\frac{|\cdot-y|^2}{\kappa t}} \right\|_{L^\theta} &\lesssim t^{-s-\frac{n}{2\theta'}} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N'} \left( t \int_{B(y, \sqrt{t})} V(x) dx \right)^s \\ &\lesssim t^{-s-\frac{n}{2\theta'}} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}. \end{aligned}$$

To prove (b) we argue as in (a), but apply the second estimate in Proposition 2.1.  $\square$

We can now show that  $T = V^s L^{-s} \in GCZO_\rho(\gamma, \theta)$  for any  $1 < \theta < \sigma/s$  and  $0 < \gamma < \sigma_0$ . Let  $K_s(x, y)$  be the kernel of  $V^s L^{-s}$ . Then

$$K_s(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty V^s(x) p_t(x, y) \frac{dt}{t^{1-s}}$$

We first prove (1.5). Now let  $B$  be a ball with  $r_B \geq 2\rho_B$  and  $y \in B(x_B, \rho_B)$ . Then by Proposition 5.3 (a), and that  $\rho(y) \sim \rho_B$ ,

$$(5.5) \quad \begin{aligned} \|K_s(\cdot, y)\|_{L^\theta(2B \setminus B)} &\lesssim \int_0^\infty \|V^s(\cdot) p_t(\cdot, y)\|_{L^\theta(2B \setminus B)} \frac{dt}{t^{1-s}} \\ &\lesssim \int_0^\infty e^{-c \frac{r_B^2}{t}} t^{-1-\frac{n}{2\theta'}} (1 + \frac{\sqrt{t}}{\rho_B})^{-N} dt =: I + II \end{aligned}$$

where

$$I = \int_0^{r_B^2} e^{-c \frac{r_B^2}{t}} t^{-1-\frac{n}{2\theta'}} (1 + \frac{\sqrt{t}}{\rho_B})^{-N} dt \lesssim r_B^{-\frac{n}{\theta'}} \left(\frac{\rho_B}{r_B}\right)^{2N}$$

and

$$II = \int_{r_B^2}^\infty e^{-c \frac{r_B^2}{t}} t^{-1-\frac{n}{2\theta'}} (1 + \frac{\sqrt{t}}{\rho_B})^{-N} dt \leq \int_{r_B^2}^\infty t^{-1-\frac{n}{2\theta'}} (1 + \frac{\sqrt{t}}{\rho_B})^{-N} dt \lesssim r_B^{-\frac{n}{\theta'}} \left(\frac{\rho_B}{r_B}\right)^{2N}$$

for any  $N > 0$ .

Let us show (1.6). Let  $B$  be any ball and  $y \in B$ . Then for each  $k \geq 1$ ,

$$\begin{aligned} \|K_s(\cdot, y) - K_s(\cdot, x_B)\|_{L^\theta(2^{k+1}B \setminus 2^k B)} &\lesssim \int_0^\infty \|V^s(\cdot) p_t(\cdot, y) - V^s(\cdot) p_t(\cdot, x_B)\|_{L^\theta(2^{k+1}B \setminus 2^k B)} \frac{dt}{t^{1-s}} \\ &= \int_0^{r_B^2} \dots + \int_{r_B^2}^\infty \dots =: I + II \end{aligned}$$

Now let for any  $0 < \gamma < \sigma_0$  we choose firstly  $\sigma_1$  such that  $\gamma < \sigma_1 < \sigma_0$ , and secondly  $\epsilon = \frac{1}{2}(\gamma + \frac{n}{\theta'})$ . Then by the triangle inequality, Proposition 5.3 (a), and the fact that  $y \in B$  we have

$$(5.6) \quad I \lesssim \int_0^{r_B^2} e^{-c 4^k \frac{r_B^2}{t}} t^{-1-\frac{n}{2\theta'}} dt \lesssim 4^{-k\epsilon} r_B^{-2\epsilon} \int_0^{r_B^2} t^{-1-\frac{n}{2\theta'}+\epsilon} dt \lesssim 4^{-k\epsilon} r_B^{-\frac{n}{\theta'}}$$

We also have by Proposition 5.3 (b)

$$II \lesssim \int_{r_B^2}^\infty e^{-c 4^k \frac{r_B^2}{t}} t^{-1-\frac{n}{2\theta'}} \left(\frac{|y - x_B|}{\sqrt{t}}\right)^{\sigma_1} dt \lesssim r_B^{\sigma_1-2\epsilon} \int_{r_B^2}^\infty t^{-1-\frac{n}{2\theta'}-\frac{\sigma_1}{2}+\epsilon} dt \lesssim 4^{-k\epsilon} r_B^{-\frac{n}{\theta'}}$$

Thus collecting our estimates for  $I$  and  $II$  we have

$$(5.7) \quad \|K_s(\cdot, y) - K_s(\cdot, x_B)\|_{L^\theta(2^{k+1}B \setminus 2^k B)} \lesssim 4^{-k\epsilon} r_B^{-\frac{n}{\theta'}} = 2^{-k\gamma} |2^k B|^{-\frac{1}{\theta'}}$$

where  $\gamma = 2\epsilon - \frac{n}{\theta'}$ .

Next we show conditions (a) and (b) of Theorem 1.2 for  $T^* = L^{-s} V^s$ . More precisely we prove

$$(5.8) \quad \log\left(\frac{\rho_B}{r_B}\right) \left( \oint_B |L^{-s} V^s 1(y) - (L^{-s} V^s 1)_B|^{\theta'} dy \right)^{1/\theta'} \leq C$$

$$(5.9) \quad \left(\frac{\rho_B}{r_B}\right)^{n(1/p-1)} \left( \oint_B |L^{-s} V^s 1(y) - (L^{-s} V^s 1)_B|^{\theta'} dy \right)^{1/\theta'} \leq C$$

for every ball  $B$  with  $r_B \leq \frac{1}{2}\rho_B$  and  $\frac{n}{n+s\sigma_0 \wedge 1} < p < 1$ . In fact, for any  $1 < \theta < \infty$ , estimates (5.8) and (5.9) are consequences of the following stronger estimate

$$(5.10) \quad |L^{-s} V^s 1(x) - L^{-s} V^s 1(y)| \lesssim \left(\frac{r_B}{\rho_B}\right)^\delta$$

for any ball  $B$  with  $r_B \leq \frac{1}{2}\rho_B$ , and  $x, y \in B$ , and any  $0 < \delta < s\sigma_0 \wedge 1$ . We shall show (5.10) by applying the following lemma.

**Lemma 5.4.** *Let  $0 < s \leq 1$ . For any  $0 < \sigma_1 < \sigma_0$ ,  $0 < \delta \leq s\sigma_0 \wedge 1$  and  $N > 0$  the following holds:*

$$(5.11) \quad |e^{-tL}V^s(x)| \leq Ct^{-s} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N}$$

for any  $x \in \mathbb{R}^n$  and  $t > 0$ , and

$$(5.12) \quad |e^{-tL}V^s(x) - e^{-tL}V^s(y)| \leq Ct^{-s} \left( \frac{\sqrt{t}}{\rho(x)} \right)^\delta \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-N} \left( \frac{|x-y|}{\sqrt{t}} \right)^{\sigma_1}$$

for all  $t > 0$  and  $|x-y| \leq \sqrt{t}$ .

*Proof.* We first prove (5.11). We have

$$\begin{aligned} e^{-tL}V^s(x) &= \int p_t(x, w)V^s(w) dw \\ &\lesssim \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-M} \int t^{-\frac{n}{2}} e^{-c\frac{|x-w|^2}{t}} V^s(w) dw \\ &\lesssim t^{-s} \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-M} \left( t \int_{B(x, \sqrt{t})} V \right)^s. \end{aligned}$$

Now from Remark 2.3 and by choosing  $M$  large enough, we obtain the required estimate.

The proof of (5.12) begins with

$$\begin{aligned} |e^{-tL}V^s(x) - e^{-tL}V^s(y)| &\leq \int |p_t(x, w) - p_t(y, w)| V^s(w) dw \\ &\lesssim \left( \frac{|x-y|}{\sqrt{t}} \right)^{\sigma_1} \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{-M} \int t^{-\frac{n}{2}} e^{-c\frac{|x-w|^2}{t}} V^s(w) dw, \end{aligned}$$

and we proceed as in (5.11). □

Let us continue with the proof of (5.10). We first write

$$\begin{aligned} |L^{-s}V^s1(x) - L^{-s}V^s1(y)| &\leq \int_0^\infty |e^{-tL}V^s(x) - e^{-tL}V^s(y)| \frac{dt}{t^{1-s}} \\ &= \int_0^{4r_B^2} + \int_{4r_B^2}^{\rho_B^2} + \int_{\rho_B^2}^\infty \cdots =: I + II + III. \end{aligned}$$

Now by (5.11), and that  $\rho(x) \sim \rho(y) \sim \rho_B$  we have for any  $0 < \delta < s\sigma_0 \wedge 1$ ,

$$I \leq \int_0^{4r_B^2} |e^{-tL}V^s(x)| \frac{dt}{t^{1-s}} + \int_0^{4r_B^2} |e^{-tL}V^s(y)| \frac{dt}{t^{1-s}} \lesssim \rho_B^{-\delta} \int_0^{4r_B^2} \frac{dt}{t^{1-\frac{\delta}{2}}} \lesssim \left( \frac{r_B}{\rho_B} \right)^\delta.$$

Now pick  $\delta < \delta_1 \leq s\sigma_0 \wedge 1$ . From  $|x-y| \leq 2r_B \leq \sqrt{t}$  and that  $\rho(x) \sim \rho_B$  we have by (5.12),

$$II \leq \int_{4r_B^2}^{\rho_B^2} t^{-s} \left( \frac{|x-y|}{\sqrt{t}} \right)^{\delta_1} \left( \frac{\sqrt{t}}{\rho_B} \right)^{\delta_1} \frac{dt}{t^{1-s}} \lesssim \left( \frac{r_B}{\rho_B} \right)^{\delta_1} \int_{4r_B^2}^{\rho_B^2} \frac{dt}{t} \lesssim \left( \frac{r_B}{\rho_B} \right)^{\delta_1} \log \left( \frac{\rho_B}{r_B} \right) \lesssim \left( \frac{r_B}{\rho_B} \right)^\delta.$$

Finally from (5.12) and by taking  $N$  large enough,

$$III \lesssim \int_{\rho_B^2}^\infty t^{-s} \left( \frac{|x-y|}{\sqrt{t}} \right)^\delta \frac{dt}{t^{1-s}} \lesssim r_B^\delta \int_{\rho_B^2}^\infty \frac{dt}{t^{1+\frac{\delta}{2}}} \lesssim \left( \frac{r_B}{\rho_B} \right)^\delta.$$

The terms  $I, II$  and  $III$  together give (5.10).

Thus (5.8) and (5.9) hold for any  $\theta > 1$ , and so we may conclude the proof of Theorem 1.7 by invoking Theorem 1.2. □

5.3.1. *The Riesz transforms  $L^{-s}V^s$ .* Before giving the proof of Theorem 1.8 we make some preliminary remarks.

Firstly, the hypothesis  $V \in RH_\infty$  ensures that  $V^s L^{-s}$  and  $L^{-s}V^s$  are both  $L^p$  bounded for all  $1 < p < \infty$ . Secondly, the conditions  $V \in RH_\infty$  and (1.9) imply

$$(5.13) \quad V(x) \leq C' \rho(x)^{-2} \quad \text{a.e. } x$$

for some  $C' > 0$ . See [24] Remark 1.8.

Our conditions on  $V$  guarantee it admits a certain smoothness, encapsulated in the following result.

**Lemma 5.5.** *If  $V$  satisfies (1.9) then for each  $0 < s \leq 1$  there exists  $C > 0$  depending only on  $s$  and  $V$  such that for every  $0 < \eta \leq 1$  we have*

$$|V^s(x) - V^s(y)| \leq \frac{C}{t^s} \left( \frac{|x-y|}{\sqrt{t}} \right)^\eta \left( \frac{\sqrt{t}}{\rho(x)} \right)^{1+2s} \left( 1 + \frac{\sqrt{t}}{\rho(x)} \right)^{2+4s}$$

whenever  $|x-y| \leq \sqrt{t}$ .

*Proof.* From the mean value theorem and part (i) of Lemma 2.2 we have, for some  $x' \in B(x, |x-y|)$ ,

$$\begin{aligned} |V^s(x) - V^s(y)| &\lesssim V^{s-1}(x') |\nabla V(x')| |x-y| \\ &\lesssim \rho(x')^{-1-2s} |x-y| \\ &\lesssim \rho(x)^{-1-2s} \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{2+4s} |x-y| \end{aligned}$$

This yields the required result if  $|x-y| \leq \sqrt{t}$ .  $\square$

This smoothness grant us the following analogues of Proposition 5.3 and Lemma 5.4 respectively.

**Proposition 5.6.** *Assume that  $V$  satisfies (1.9). Then for each  $0 < s \leq 1$ , there exists  $C > 0$  such that the following holds for all  $N > 0$ ,*

(a) *For every  $x, y \in \mathbb{R}^n$ ,  $t > 0$ ,*

$$|V^s(y) p_t(x, y)| \leq C t^{-s-\frac{n}{2}} e^{-\frac{|x-y|^2}{ct}} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}.$$

(b) *For all  $|y-y'| \leq \sqrt{t}$  and any  $0 < \eta \leq 1$  we have*

$$|V^s(y) p_t(x, y) - V^s(y') p_t(x, y')| \leq C t^{-s-\frac{n}{2}} e^{-\frac{|x-y|^2}{ct}} \left( \frac{|y-y'|}{\sqrt{t}} \right)^\eta$$

*Proof of Proposition 5.6.* To prove (a) we observe that from the heat kernel bounds in Proposition 2.1 and from (5.13) that

$$|V^s(y) p_t(x, y)| \lesssim t^{-s-\frac{n}{2}} e^{-\frac{|x-y|^2}{ct}} \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N'} \left( \frac{\sqrt{t}}{\rho(y)} \right)^{2s}$$

The result now follows by taking  $N'$  large enough.

For part (b) we write

$$\begin{aligned} |V^s(y) p_t(x, y) - V^s(y') p_t(x, y')| &\leq V^s(y) |p_t(x, y) - p_t(x, y')| + |V^s(y) - V^s(y')| |p_t(x, y')| \\ &=: I + II \end{aligned}$$

From the second estimate in Proposition 2.1 and (5.13) we have

$$I \lesssim \rho(y)^{-2s} t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{ct}} \left( \frac{|y-y'|}{\sqrt{t}} \right)^\eta \left( 1 + \frac{\sqrt{t}}{\rho(y)} \right)^{-N'} \lesssim t^{-s-\frac{n}{2}} e^{-\frac{|x-y|^2}{ct}} \left( \frac{|y-y'|}{\sqrt{t}} \right)^\eta$$

by taking  $N'$  large enough. Next we have from the bounds on the heat kernel, that  $|y - y'| \leq \sqrt{t}$ , and Lemma 5.5,

$$\begin{aligned} II &\lesssim |V^s(y) - V^s(y')| t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{ct}} \left(1 + \frac{\sqrt{t}}{\rho(y')}\right)^{-N'} \\ &\lesssim t^{-s-\frac{n}{2}} e^{-\frac{|x-y|^2}{ct}} \left(\frac{|y-y'|}{\sqrt{t}}\right)^\eta \left(\frac{\sqrt{t}}{\rho(y')}\right)^{1+2s} \left(1 + \frac{\sqrt{t}}{\rho(y')}\right)^{2+4s} \left(1 + \frac{\sqrt{t}}{\rho(y')}\right)^{-N'} \end{aligned}$$

which gives the required estimate after taking  $N'$  large enough.  $\square$

**Lemma 5.7.** *Suppose that  $V$  satisfies (1.9) and  $0 < s \leq 1$ . Then for any  $0 < \delta \leq 2s \wedge 1$  and  $0 < \eta \leq 1$  the following holds:*

$$(5.14) \quad |V^s(x)e^{-tL}1(x)| \leq Ct^{-s} \left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N}$$

for any  $x \in \mathbb{R}^n$  and  $t > 0$ , and

$$(5.15) \quad |V^s(x)e^{-tL}1(x) - V^s(y)e^{-tL}1(y)| \leq Ct^{-s} \left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} \left(\frac{|x-y|}{\sqrt{t}}\right)^\eta$$

for all  $t > 0$  and  $|x - y| \leq \sqrt{t}$ .

*Proof of Lemma 5.7.* Firstly by (5.13) and the bounds on the heat kernel,

$$|V^s(x)e^{-tL}1(x)| \lesssim \rho(x)^{-2s} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N'}$$

Thus (5.14) follows by considering the cases  $\sqrt{t} \geq \rho(x)$  and  $\sqrt{t} < \rho(x)$  and taking suitable  $N'$ . Turning to (5.15) we write

$$|V^s(x)e^{-tL}1(x) - V^s(y)e^{-tL}1(y)| \leq |V^s(x) - V^s(y)| |e^{-tL}1(x)| + V^s(y) |e^{-tL}1(x) - e^{-tL}1(y)|$$

Now from Lemma 5.5 we have

$$|V^s(x) - V^s(y)| |e^{-tL}1(x)| \lesssim t^{-s} \left(\frac{|x-y|}{\sqrt{t}}\right)^\eta \left(\frac{\sqrt{t}}{\rho(x)}\right)^{1+2s} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{2+4s} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N'}$$

which gives the right hand side of (5.15). Next from (2.2), (5.13), and Lemma 2.2 (i),

$$\begin{aligned} V^s(y) |e^{-tL}1(x) - e^{-tL}1(y)| &\leq V^s(y) \int |p_t(x, w) - p_t(y, w)| dw \\ &\lesssim \rho(y)^{-2s} \left(\frac{|x-y|}{\sqrt{t}}\right)^\eta \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N'} \\ &\lesssim \rho(x)^{-2s} \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{4s} \left(\frac{|x-y|}{\sqrt{t}}\right)^\eta \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N'} \end{aligned}$$

which also yields the right hand side of (5.15).  $\square$

**Proof of Theorem 1.8.** We shall show that  $T = L^{-s}V^s \in GCZO_\rho(\gamma, \theta)$  for any  $1 < \theta < \infty$  and  $0 < \gamma < 1$ . Note firstly that  $V \in RH_\infty$  implies that  $L^{-s}V^s$  is bounded on  $L^\theta$  for any  $1 < \theta < \infty$ . Next we set

$$K_s^*(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty p_t(x, y) V^s(y) \frac{dt}{t^{1-s}}$$

to be the kernel of  $L^{-s}V^s$ .

Let us show (1.5). Fix a ball  $B$  with  $r_B \geq 2\rho_B$  and  $y \in B$ . Then we have  $\rho(y) \sim \rho_B$ . Thus from Proposition 5.6, for any  $1 < \theta < \infty$ ,

$$\|K_s^*(\cdot, y)\|_{L^\theta(2B \setminus B)} \lesssim \int_0^\infty \|p_t(\cdot, y) V^s(y)\|_{L^\theta(2B \setminus B)} \frac{dt}{t^{1-s}} \lesssim \int_0^\infty e^{-c\frac{r_B^2}{t}} t^{-1-\frac{n}{2\theta'}} \left(1 + \frac{\sqrt{t}}{\rho_B}\right)^{-N} dt$$



At this point we can continue as in (5.5).

Let us turn to (1.6). Now for each  $1 < \theta < \infty$  and  $0 < \gamma < 1$  let us take  $\epsilon = \frac{1}{2}(\gamma + \frac{n}{\theta})$ . Let  $B$  be any ball and  $y \in B$ . Then for each  $k \geq 1$ ,

$$\begin{aligned} \|K_s^*(\cdot, y) - K_s^*(\cdot, x_B)\|_{L^\theta(2^{k+1}B \setminus 2^k B)} &\lesssim \int_0^\infty \|p_t(\cdot, y)V^s(y) - p_t(\cdot, x_B)V^s(x_B)\|_{L^\theta(2^{k+1}B \setminus 2^k B)} \frac{dt}{t^{1-s}} \\ &= \int_0^{r_B^2} \cdots + \int_{r_B^2}^\infty \cdots =: I + II \end{aligned}$$

We can apply Proposition 5.6 (a) and proceed as in (5.6) to obtain

$$I \lesssim 4^{-k\epsilon} r_B^{-\frac{n}{\theta'}}$$

For the second term, Proposition 5.6 (b) gives

$$II \lesssim \int_{r_B^2}^\infty e^{-c4^k \frac{r_B^2}{t}} t^{-1-\frac{n}{2\theta'}} \left( \frac{|y - x_B|}{\sqrt{t}} \right) dt \lesssim 4^{-k\epsilon} r_B^{-\frac{n}{\theta'}}$$

Combining our estimates for  $I$  and  $II$  gives (1.6) because  $\gamma = 2\epsilon - \frac{n}{\theta'}$ .

Next we prove that  $T^* = V^s L^{-s}$  satisfies (a) and (b) of Theorem 1.2. As before this follows from the following version of (5.10): for each ball  $B$  with  $r_B \leq \frac{1}{2}\rho_B$ ,

$$(5.16) \quad |V^s L^{-s} 1(x) - V^s L^{-s} 1(y)| \lesssim \left( \frac{r_B}{\rho_B} \right)^\delta$$

for any  $x, y \in B$  and  $0 < \delta < 2s \wedge 1$ . We can obtain (5.16) by arguing as in (5.10), but using Lemma 5.7 in place of Lemma 5.4.  $\square$

**Acknowledgments.** T.A. Bui is supported by ARC DP 140100649.

#### REFERENCES

- [1] P. Auscher, X.T. Duong, and A. McIntosh, *Boundedness of Banach space valued singular integral operators and Hardy spaces*, unpublished preprint (2005).
- [2] J.J. Betancor, R. Crescimbeni, J.C. Farina, P.R. Stinga and J.L. Torrea, *A T1 criterion for Hermite-Calderón-Zygmund operators on the  $BMO_H(\mathbb{R}^n)$  space and applications*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), no. 1, 157–187.
- [3] B. Bongioanni, E. Harboure, and O. Salinas, *Weighted inequalities for negative powers of Schrödinger operators*, J. Math. Anal. Appl. **348** (2008), 12–27.
- [4] ———, *Weighted inequalities for commutators of Schrödinger-Riesz transforms*, J. Math. Anal. Appl. **392** (2012), no. 1, 6–22.
- [5] S. Campanato, *Proprieta di holderianita di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 175–188.
- [6] G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. **120** (1984), 371–397.
- [7] X.T. Duong and L.X. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. **18** (2005), 943–973.
- [8] ———, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. **58** (2005), no. 10, 1375–1420.
- [9] J. Dziubański, G. Garrigós, T. Martínez, J. L. Torrea, and J. Zienkiewicz, *BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Math. Z. **249** (2005), no. 2, 329–356.
- [10] J. Dziubański and J. Zienkiewicz, *Hardy space  $H^1$  associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoamericana **15** (1999), no. 2, 279–296.
- [11] ———,  *$H^p$  spaces for Schrödinger operators*, Fourier analysis and related topics (Bedlewo, 2000), Banach Center Publ., vol. 56, Polish Acad. Sci., Warsaw, 2002, 45–53.
- [12] ———,  *$H^p$  spaces associated with Schrödinger operators with potentials from reverse Hölder classes*, Colloq. Math. **98** (2003), no. 1, 5–38.
- [13] L. Grafakos, “Classical and Modern Fourier Analysis”, Pearson Education, Inc., Upper Saddle River, NJ, 2004.

- [14] Z. Guo, P. Li and L. Peng,  $L^p$  boundedness of commutators of Riesz transforms associated to Schrödinger operator, J. Math. Anal. Appl. **341** (2008), no. 1, 421–432.
- [15] Y.S. Han, Y.C. Han and J. Li, *Criterion of the boundedness of singular integrals on spaces of homogeneous type*, submitted.
- [16] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and L. Yan, *Hardy spaces associated with non-negative self adjoint operators satisfying Davies–Gaffney estimates*, Mem. Amer. Math. Soc. **214** (2011), no. 1007.
- [17] R. Jiang and Da. Yang, *Orlicz–Hardy spaces associated with operators satisfying Davies–Gaffney estimates*, Comm. Contemp. Math. **13** (2011), no.2, 331–373.
- [18] R. Jiang, Da. Yang and Do. Yang, *Maximal function characterizations of Hardy spaces associated with magnetic Schrödinger operators*, Forum Math. **24** (2012), no. 3, 471–494.
- [19] L.D. Ky, *On weak\*-convergence in  $H_L^1(\mathbb{R}^d)$* , Potential Anal. **39** (2013), no. 4, 355–368.
- [20] F. K. Ly, *Second order Riesz transforms associated to the Schrödinger operator for  $p \leq 1$* , J. Math. Anal. App. **410** (2014), no. 1, 391–402.
- [21] ———, *Classes of weights and second order Riesz transforms associated to Schrödinger operators*, J. Math. Soc. Japan., to appear.
- [22] T. Ma, P. R. Stinga, J. Torrea, and C. Zhang, *Regularity estimates in Hölder spaces for Schrödinger operators via a  $T1$  theorem*, Ann. Mat. Pura. Appl. (4) **193** (2014), no. 2, 561–589.
- [23] Z. Shen,  *$L^p$  estimates for Schrödinger operators with certain potentials*, Ann. Inst. Fourier (Grenoble) **45** (1995), 513–546.
- [24] ———, *Estimates in  $L^p$  for magnetic Schrödinger operators*, Indiana Univ. Math. J. **45** (1996), no. 3, 817–841.
- [25] S. Sugano, *Estimates for the Operators  $V^\alpha(-\Delta + V)^{-\beta}$  and  $V^\alpha \nabla(-\Delta + V)^{-\beta}$  with Certain Non-negative Potentials  $V$* , Tokyo J. Math. **21** (1998), no. 2, 441–452.
- [26] D. Yang and S. Yang, *Weighted local Orlicz Hardy spaces with applications to pseudo-differential operators*, Dissertationes Math. (Rozprawy Mat.) **478** (2011), 78 pp.
- [27] ———, *Local Hardy spaces of Musielak-Orlicz type and their applications*, Sci. China Math. **55** (2012), no. 8, 1677–1720.
- [28] Da. Yang and Do. Yang, *Hardy spaces associated with magnetic Schrödinger operators on strongly Lipschitz domains*, Nonlinear Anal. **75** (2012), no. 18, 6433–6447.
- [29] ———, *Maximal function characterizations of Musielak-Orlicz-Hardy spaces associated with magnetic Schrödinger operators*, Front. Math. China **10** (2015), no. 5, 1203–1232.
- [30] Da. Yang, Do. Yang and Y. Zhou, *Localized Morrey-Campanato spaces on metric measure spaces and applications to Schrödinger operators*, Nagoya Math. J. **198** (2010), 77–119.
- [31] ———, *Localized BMO and BLO spaces on RD-spaces and applications to Schrödinger operators*, Commun. Pure Appl. Anal. **9** (2010), no. 3, 779–812.
- [32] D. Yang and Y. Zhou, *Localized Hardy spaces  $H^1$  related to admissible functions on RD-spaces and applications to Schrödinger operators*, Trans. Amer. Math. Soc. **363** (2011), 1197–1239.
- [33] J. Zhong, “Harmonic analysis for some Schrödinger type operators”, Ph. D. Thesis, Princeton University, 1993.

DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2109, AUSTRALIA  
*E-mail address*: the.bui@mq.edu.au

DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2109, AUSTRALIA  
*E-mail address*: ji.li@mq.edu.au

SYDNEY INSTITUTE OF BUSINESS AND TECHNOLOGY, LEVEL 2, 11 YORK STREET, NSW 2000, AUSTRALIA  
*E-mail address*: ken.ly@learning.sibt.nsw.edu.au